Finding a subdivision of a prescribed digraph of order 4

Frédéric Havet, Ana Karolinna Maia de Oliveira, Bojan Mohar

Finding a subdivision of a prescribed digraph of order 4

Frédéric Havet , A. Karolinna Maia, Bojan Mohar

RESEARCH REPORT
N° 8773
September 2015
Project-Teams Coati

ISSN 0249-6399   ISRN INRIA/RR--8773--FR+ENG

Finding a subdivision of a prescribed digraph of order 4

Frédéric Havet *, A. Karolinna Maia†, Bojan Mohar‡

Project-Teams Coati

Research Report n° 8773 — September 2015 — 53 pages

Abstract: The problem of when a given digraph contains a subdivision of a fixed digraph F is considered. Bang-Jensen et al. [2] laid out foundations for approaching this problem from the algorithmic point of view. In this paper we give further support to several open conjectures and speculations about algorithmic complexity of finding F-subdivisions. In particular, up to 5 exceptions, we completely classify for which 4-vertex digraphs F, the F-subdivision problem is polynomial-time solvable and for which it is NP-complete. While all NP-hardness proofs are made by reduction from some version of the 2-linkage problem in digraphs, some of the polynomial-time solvable cases involve relatively complicated algorithms.

Key-words: digraph, oriented graph, subdivision, linkage, polynomial-time algorithm, NP-completeness

* COATI Project, I3S (CNRS/UNSA) & INRIA. email: `frederic.havet@cnrs.fr` Partially supported by the ANR project STINT. This work was mainly done when F. Havet was on leave at Simon Fraser University, PIMS, UMI 3069, CNRS.

† ParGO research group, Department of Computer Science, Federal University of Ceará, Brazil. email: `karol@lia.ufc.br` Partly supported by CAPES/Brazil.

‡ Department of Mathematics, Simon Fraser University, Canada. On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Slovenia. email: `mohar@sfu.ca` Supported in part by the Research Grant P1–0297 of ARRS (Slovenia), by an NSERC Discovery Grant (Canada) and by the Canada Research Chair program.

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Trouver use subdivision d'un digraph donné d'ordre 4.

Résumé : Nous considérons le problème consistant à déterminer si un digraphe donné contient une subdivision d'un digraphie F fixé. Bang-Jensen et al. [2] ont posé les bases d'une approche algorithmique de ce problème. Dans ce rapport, nous poursuivons leur étude et donnons des indications sur la validité de certaines conjectures et spéculations. En particulier, à 5 exceptions près, nous caractérisons les digraphes à 4 sommets : ceux pour lequel le problème est NP-complet et ceux pour lequel il est polynomial. Alors que toutes les preuves de NP-complétude sont des réductions à une version particulière du problème du 2-linkage dans les digraphes, certains cas solubles en temps polynomial nécessitent des algorithmes relativement compliqués.

Mots-clés : digrahe, graphe orienté, subdivision, linkage, algorithme polynomial, NP-complétude

1 Introduction

In this paper, all digraphs are meant to be *strict*, that is without loops and without multiple arcs. In one occasion, however, multiple arcs will be allowed. In that case, we will use the term *multidigraph*. We follow standard terminology as used in [1, 4].

A *subdivision of a digraph* F, also called an F*-subdivision*, is a digraph obtained from F by replacing each arc ab of F by a directed (a, b)-path. In this paper, we consider the following problem for a fixed digraph F.

F-SUBDIVISION
Input: A digraph D.
Question: Does D contain a subdivision of F?

Bang-Jensen et al. [2] conjectured that there is a dichotomy between NP-complete and polynomial-time solvable instances.

Conjecture 1. For every digraph F, the F-SUBDIVISION problem is polynomial-time solvable or NP-complete.

According to this conjecture, there are only two kinds of digraphs F: *hard* digraphs F, for which F-SUBDIVISION is NP-complete, and *tractable* digraphs, for which F-SUBDIVISION is solvable in polynomial-time.

Bang-Jensen et al. [2] proved that many digraphs are hard; see Theorem 9 in Section 4. In particular, every digraph in which every vertex v is *big* (that is such that either $d^+(v) \geq 3$, or $d^-(v) \geq 3$, or $d^-(v) = d^+(v) = 2$) is hard. They also give many examples of tractable digraphs. See Subsection 2.4. However, there is no clear evidence, of which graph should be tractable and which one should be hard, despite some results and conjectures give some outline.

Establishing a conjecture of of Johnson et al. [9], Kawabarayashi and Kreutzer [10] proved the Directed Grid Theorem.

Theorem 2 (Kawabarayashi and Kreutzer [10])**.** *For any positive integer k, there exists an integer $f(k)$ such that every digraph with directed treewidth greater than $f(k)$ contains a cylindrical grid of order k as a butterfly minor.*

Here, a *cylindrical grid* of order k consists of k concentric directed cycles and $2k$ directed paths connecting the cycles in alternating directions. See Figure 1 for an illustration. A *butterfly minor* of a digraph D is a digraph obtained from a subgraph of D by contracting arcs which are either the only outgoing arc of their tail or the only incoming arc of their head.

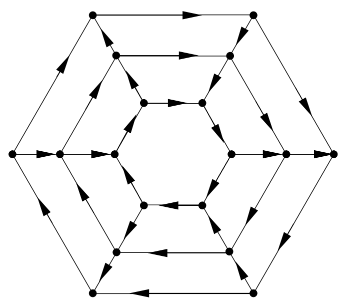
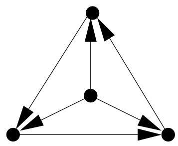


Figure 1: The cylindrical grid of order 3.



W_3

Figure 2: The 3-wheel W_3.

Because the k-LINKAGE problem (See Subsection 2.3 for definitions and details on k-LINKAGE) is polynomial-time solvable on digraph with bounded directed treewidth (see [2] for more details.), this directly implies the following :

Corollary 3. *F-SUBDIVISION is polynomial-time solvable when F is a planar digraph with no big vertices.*

Indeed, for every planar digraph F with no big vertices there is an F-subdivision in a cylindrical grid of order k, for some sufficiently large k. Therefore one can solve F-SUBDIVISION as follows. Given a digraph G, one estimates its directed treewidth. If it is greater than $f(k)$ (as defined in Theorem 2), then it contains a cylindrical grid of order k and so an F-subdivision and we return 'Yes'. Otherwise it has small treewidth, and one can solve F-subdivision in polynomial time.

Before the Directed Grid Theorem was proved, we found elementary proofs (i.e. not using this theorem) to show that digraphs of order 4 with no big vertices (all digraphs of order 4 are planar) are easy. The proofs can be easily implemented and the derived algorithms are certainly of lower complexity than the ones derived from the Directed Grid Theorem. They can be found in Appendix A.

On the other hand, Bang-Jensen et al. [2] proposed the following sort of counterpart.

Conjecture 4 (Bang-Jensen et al. [2]). F-SUBDIVISION is NP-complete for every non-planar digraph F.

Bang-Jensen et al. [2] were able to classify all digraphs of order at most 3: they are all tractable except the complete symmetric digraph on three vertices, which is hard. In this paper, we aim at classifying all digraphs of order 4.

We first deal in Section 3 with oriented graphs. We prove that all oriented graphs of order 4 are tractable. In particular, we show in Subsection 3.1 that the wheel W_3 is tractable. The *wheel* W_k is the graph obtained from the directed cycle on k vertices $\vec{C}_k$ by adding a vertex, called the *centre*, dominating every vertex of $\vec{C}_k$. In [2], Bang-Jensen et al. proved that W_2 is tractable and that W_k-SUBDIVISION is NP-complete for all $k \geq 4$. The case of W_3 was left as an open problem.

This also completes the classification of tournaments. Bang-Jensen et al. [2] proved that every tournament of order at most 3 is tractable, and that every tournament of order at least 5 is hard (see Theorem 9). They also show that the transitive tournament of order 4 is tractable. The other tournaments of order four are W_3, its converse, and ST_4, the strong tournament of order 4, no vertex of which is big.

Next, we turn to digraphs of order 4, which are not oriented graphs. In Section 4, we show some of them to be hard, and in Section 5, we prove some of them to be tractable. Using these results, we are able to classify all digraphs of order 4 except for five of them (up to directional duality). These are the digraphs O_i for $1 \leq i \leq 5$ depicted Figure 3.

1.1 Finding an F-subdivision

The letters n and m will always denote the number of vertices and arcs of the input digraph D of the problem in question. By *linear time*, we mean $O(n + m)$ time.

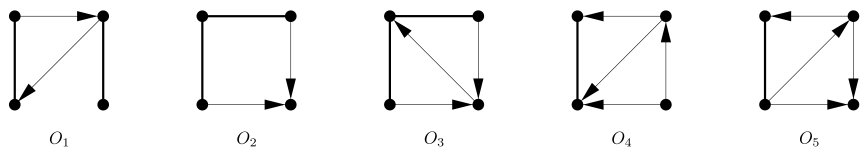


Figure 3: Digraphs on 4-vertices that are not known to be tractable or hard. Bold undirected edges represent directed 2-cycles.

Lemma 5. *If F-*SUBDIVISION *can be solved in $f(n,m)$ time, where f is non-decreasing in m, then there is an algorithm that finds an F-subdivision (if one exists) in a digraph in $((m+1)\cdot f(n,m)+m)$ time.*

Proof. Suppose that there exists an algorithm $F\text{-}\mathtt{decide}(D)$ that decides in $f(n,m)$ whether D contains an F-subdivision. We now construct an algorithm $F\text{-}\mathtt{find}(D)$ that finds an F-subdivision in D if there is one, and returns 'no' otherwise. It proceeds as follows.

Let $a_1,\dots,a_m$ be the arcs of D. If $F\text{-}\mathtt{decide}(D)$ returns 'no', then we also return 'no'. If not, then D contains an F-subdivision, we find it as follows: We initialize $D_0 := D$. For $i = 1$ to m, $D_i := D_{i-1}\setminus a_i$ if $F\text{-}\mathtt{decide}(D_{i-1}\setminus a_i)$ returns 'yes', and $D_i := D_{i-1}$ otherwise.

$F\text{-}\mathtt{find}$ is valid because at step i, we delete the arc a_i if and only if there is an F-subdivision not containing i. Hence at each step i, we are sure that D_i contains an F-subdivision, and that any F-subdivision must contain all the arcs of $A(D_i)\cap\{a_1,\dots,a_i\}$.

$F\text{-}\mathtt{find}$ runs $(m+1)$ times the algorithm $F\text{-}\mathtt{decide}$ and removes at most m times an arc. Therefore, it runs in time $(m+1)\cdot f(n,m)+m$. □

Lemma 5 implies that deciding if there is an F-subdivision in a digraph is polynomial-time solvable, if and only if, finding an F-subdivision in a digraph is polynomial-time solvable. Therefore, since we are primarily interested in determining if the problems are polynomial-time solvable or NP-complete, and for sake of clarity, we only present algorithms for solving F-SUBDIVISION as a decision problem. However, the proofs of validity of all given algorithms always rely on constructive claims. Hence each algorithm can be easily transformed into a polynomial-time algorithm for finding an F-subdivision in a given digraph. Moreover, the reader can check that the additional work does not increase the time complexity. Hence, our algorithms for finding F-subdivisions have the same complexity as their decision versions.

2 Preliminaries

2.1 Definitions and notations

We rely on [1, 4] for standard notation and concepts. Let D be a digraph. The *converse* of D is the digraph $\overline{D}$ obtained from D by reversing the orientation of all arcs. We denote by $UG(D)$ the underlying (multi)graph of D, that is, the (multi)graph we obtain by replacing each arc by an edge. To every graph G, we can *associate* a symmetric digraph by replacing every edge uv by the two arcs uv and vu.

A *source* in D is a vertex of indegree zero and a *sink* is a vertex of outdegree zero.

An *oriented graph* is an orientation of an undirected graph. In other words, it is a digraph with no directed cycles of length 2. An *oriented path* is an orientation of an undirected path. Hence an oriented path P is a sequence $(x_1,a_1,x_2,a_2,\dots,a_{n-1},x_n)$, where the x_i are distinct vertices and for all $1\le j\le n-1$, a_j is either the arc x_jx_{j+1} or the arc $x_{j+1}x_j$. For sake of clarity, we often refer to such an oriented path P by the underlying undirected path $x_1x_2\dots x_n$. This is a slight abuse, because the oriented path P is not completely determined by this sequence as there are two possible orientations for each edge. However, when we use this notation, either the orientation does not matter or it is clear from the context.

Let $P = x_1x_2 \cdots x_n$ be an oriented path. We say that P is an (x_1, x_n)*-path*. The vertex x_1 is the *initial vertex* of P and x_n its *terminal vertex*. We denote the initial vertex of P by $s(P)$ and the terminal vertex of P by $t(P)$. The subpath $x_2 \cdots x_{n-1}$ is denoted by P°. If x_1x_2 is an arc, then P is an *outpath*, otherwise P is an *inpath*. The path P is *directed* if no vertex is the tail of two arcs in P nor the head of two arcs. In other words, all arcs are oriented in the same direction. There are two kinds of directed paths, namely directed outpaths and directed inpaths. For convenience, a directed outpath is called a *dipath*. The *blocks* of an oriented path P are the maximal directed subpaths of P. We often enumerate them from the initial vertex to the terminal vertex of the path. The number of blocks of P is denoted by $b(P)$. The *opposite path* of P, denoted $\overleftarrow{P}$, is the path $x_nx_{n-1} \cdots x_1$. For $1 \leq i \leq j \leq n$, we denote by $P[x_i, x_j]$ (resp. $P]x_i, x_j[$, $P]x_i, x_j]$, $P[x_i, x_j[$), the oriented subpath $x_ix_{i+1} \dots x_j$ (resp. $x_{i+1}x_{i+2} \dots x_{j-1}$, $x_{i+1}x_{i+2} \dots x_j$, $x_ix_{i+1} \dots x_{j-1}$).

The above definitions and notation can also be used for oriented cycles. If $C = x_1x_2 \dots x_nx_1$ is an oriented cycle, we shall assume that either C is a *directed cycle*, that is x_ix_{i+1} is an arc for all $1 \leq i \leq n$, where $x_{n+1} = x_1$, or both edges of C incident with x_1 are directed outwards, i.e. x_1x_2 and x_1x_n are arcs of C.

For a set X of vertices, the *outsection* of X in D, denoted by $S^+_D(X)$, is the set of vertices that are reachable from X by a dipath. The outsection of a set in a digraph can be found in linear time using the Breadth-First Search. The directional dual notion, the *insection* of X in D is denoted by $S^-_D(X)$.

The digraph D is *connected* (resp. k*-connected*) if $UG(D)$ is a connected (resp. k-connected) graph. It is *strongly connected*, or *strong*, if for any two vertices u, v, there is a (u, v)-dipath in D. If D is strong, we use the notation $D[u, v]$ to denote any (u, v)-dipath in D. The disjoint union of two digraphs D_1 and D_2 is denoted $D_1 + D_2$.

By *contracting* a vertex-set $X \subseteq V(D)$ we refer to the operation of first taking the digraph $D - X$ and then adding new vertex v_X and adding the arc v_Xw for each $w \in V(D - X)$ with an inneighbour in X and the arc uv_X for each $u \in V(D - X)$ with an outneighbour in X. The *contraction* of a non-strong digraph D is the digraph obtained by contracting all strong components of D.

2.2 Menger's Theorem

Let X and Y be two sets of vertices in a digraph D. An (X, Y)*-dipath* is a dipath with initial vertex in X, terminal vertex in Y and all internal vertices in $V(D) \setminus (X \cup Y)$. For notational clarity, for a vertex x (resp. a subdigraph S of D), we abbreviate $\{x\}$ to x (resp. $V(S)$ to S) in the notation. For example, an (x, S)-dipath is an $(\{x\}, V(S))$-dipath.

Let D be a digraph, and let x and y be distinct vertices of D. Two (x, y)-paths P and Q are *internally disjoint* if they have no internal vertices in common, that is if $V(P) \cap V(Q) = \{x, y\}$. A k*-separation* of (x, y) in D is a partition (W, S, Z) of its vertex set such that $x \in W$, $y \in Z$, $|S| \leq k$, each vertex in W can be reached from x by a dipath in $D[W]$, and there is no arc from W to Z.

One version of the celebrated Menger's Theorem is the following.

Theorem 6 (Menger). *Let k be a positive integer, let D be a digraph, and let x and y be distinct vertices in D such that $xy \notin A(D)$. Then, in D, either there are $k + 1$ pairwise internally disjoint (x, y)-dipaths, or there is a k-separation of (x, y).*

For any fixed k, there exist algorithms running in linear time that, given a digraph D and two distinct vertices x and y such that $xy \notin A(D)$, returns either $k + 1$ internally disjoint (x, y)-dipaths in D, or a k-separation (W, S, Z) of (x, y). Indeed, in such a particular case, any flow algorithm like Ford–Fulkerson algorithm for example, performs at most $k + 1$ incrementing-path searches, because it increments the flow by 1 each time, and we stop when the flow has value $k + 1$, or if we find a cut of size less than

$k+1$, which corresponds to a k-separation. Moreover each incrementing-path search consists in a search (usually Breadth-First Search) in an auxiliary digraph of the same size, and so is done in linear time. For more details, we refer the reader to the book of Ford and Fulkerson [6] or Chapter 7 of [4]. We call such an algorithm a *Menger algorithm*.

Observe that using Menger algorithms, one can decide if there are k internally disjoint (x, y)-dipaths in a digraph D. If $xy \notin A(D)$, then we apply a Menger algorithm directly; if $xy \in A(D)$, then we check whether there are $k-1$ internally disjoint (x, y)-dipaths in $D \setminus xy$.

Let D be a digraph. Let X and Y be non-empty sets of vertices in D. Two (X, Y)-paths P and Q are *disjoint* if they have no vertices in common, that is if $V(P) \cap V(Q) = \emptyset$. A *$k$-separation* of (X, Y) in D is a partition (W, S, Z) of its vertex set such that $X \subseteq W \cup S$, $Y \subseteq Z \cup S$, $|S| \leq k$, all vertices of W can be reached from $X \setminus S$ by dipaths in $D[W]$, and there is no arc from W to Z.

Let x be a vertex of D and Y be a non-empty subset of $V(D) \setminus \{x\}$. Two (x, Y)-paths P and Q are *independent* if $V(P) \cap V(Q) = \{x\}$. A *k-separation* of (x, Y) in D is a partition (W, S, Z) of its vertex set such that $x \in W$, $Y \subseteq Z \cup S$, $|S| \leq k$, all vertices of W can be reached from x by dipaths in $D[W]$, and there is no arc from W to Z.

Let y be a vertex of D and X be a non-empty subset of $V(D)\setminus\{y\}$. Two (X, y)-paths are *independent* if $V(P) \cap V(Q) = \{y\}$. A *k-separation* of (X, y) in D is a partition (W, S, Z) of its vertex set such that W and Z are non-empty, $X \subseteq W \cup S$, $y \in Z$, $|S| \leq k$, all vertices of W can be reached from $X \setminus S$ by dipaths in $D[W]$, and there are no arcs from W to Z.

Let $W \subset V(D)$. The digraph D_W is the one obtained from D by adding a vertex s_W and the arcs $s_W w$ for all $w \in W$ and the digraph D^W is the one obtained from D by adding a vertex t_W and the arcs wt_W for all $w \in W$.

Applying Theorem 6 to D_X^Y and (s_X, t_Y) (resp. D^Y and (x, t_Y), D_X and (s_X, y)), we obtain the following version of Menger's Theorem.

Theorem 7 (Menger). *Let k be a positive integer, and let D be a digraph. Then the following hold.*

(i) If X and Y are two non-empty subsets of $V(D)$, then, in D, either there are $k+1$ pairwise disjoint (X, Y)-dipaths, or there is a k-separation of (X, Y).

(ii) If x is a vertex of D and Y is a non-empty subset of $V(D)$, then, in D, either there are $k+1$ pairwise independent (x, Y)-dipaths in D, or there is a k-separation of (x, Y).

(iii) If X is a non-empty subset of $V(D)$ and y is a vertex of D and , then, in D, either there are $k+1$ pairwise independent (X, y)-dipaths in D, or there is a k-separation of (X, y).

Moreover, a Menger Algorithm applied to D_X^Y and (s_X, t_Y) (resp. D^Y and (x, t_Y), D_X and (s_X, Y)) finds in linear time the $k+1$ dipaths or the separation as described in Theorem 7 (i) (resp. (ii), (iii)).

Let x and y be two vertices. An (x, y)-*handle* is an (x, y)-dipath if $x \neq y$, and a directed cycle containing x if $x = y$. Let $y_1, \ldots, y_p$ be p distinct vertices, $k_1, \ldots, k_p$ be positive integers and set $k = k_1 + \cdots + \cdots k_p$. One can decide if there are k internally disjoint handles $P_1, \ldots, P_k$ such that k_i of them are (x, y_i)-handles, $1 \leq i \leq p$, by applying a Menger algorithm between in an auxiliary digraph D'. This digraph is obtained from $D - (\{y_1, \ldots, y_k\} \setminus \{x\})$ as follows. Add a new vertex y. For each i, create a set B_i of k_i new vertices and all arcs from $N_D^-(y_i)$ to B_i and from y_i to y.

Similarly, suppose that X is a set of vertices, $y_1, \ldots, y_p$ be p distinct vertices not in X, and $k = k_1 + \cdots + \cdots k_p$. One can decide if there are k internally disjoint paths $P_1, \ldots, P_k$, all with distinct initial vertices in X, and such that k_i of them are terminating in y_i, $1 \leq i \leq p$.

2.3 Linkage and disjoint directed cycles in digraphs

Let $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$ be distinct vertices of a digraph D. A *k-linkage* from $(x_1, x_2, \ldots, x_k)$ to $(y_1, y_2, \ldots, y_k)$ in D is a system of disjoint dipaths $P_1, P_2, \ldots, P_k$ such that P_i is an (x_i, y_i)-path in D. Fortune, Hopcroft and Wyllie [7] showed that for any $k \geq 2$, k-LINKAGE is NP-complete. The problem is also NP-complete when restricted to some classes of digraphs. Recall that a vertex v is *big* if either $d^+(v) \geq 3$, or $d^-(v) \geq 3$, or $d^-(v) = d^+(v) = 2$.

RESTRICTED 2-LINKAGE
Input: A digraph D without big vertices in which x_1 and x_2 are sources and y_1 and y_2 are sinks.
Question: Is there a 2-linkage from (x_1, x_2) to (y_1, y_2) in D?

Theorem 8 (Bang-Jensen et al. [2]). *The* RESTRICTED 2-LINKAGE *problem is NP-complete.*

Using this theorem, Bang-Jensen et al. [2] deduced a sufficient condition for F-SUBDIVISION to be NP-complete.

For a digraph D, we denote by $B(D)$ the set of its big vertices. A *big path* in a digraph is a directed path whose endvertices are big and whose internal vertices all have both indegree and outdegree equal to 1 in D (in particular an arc between two big vertices is a big path). Note also that two big paths with the same endvertices are necessarily internally disjoint.

The *big paths digraph* of D, denoted $BP(D)$, is the multidigraph with vertex set $V(D)$ in which there are as many arcs between two vertices u and v as there are big (u, v)-paths in D.

Theorem 9 (Bang-Jensen et al. [2]). *Let F be a digraph. If F contains two arcs ab and cd whose endvertices are big vertices and such that $(BP(F) \setminus \{ab, cd\}) \cup \{ad, cb\}$ is not isomorphic to $BP(F)$, then F-*SUBDIVISION *is NP-complete.*

Corollary 10. *Let F be a digraph. If F contains a directed cycle of length 2 whose vertices are big, then F-*SUBDIVISION *is NP-complete.*

So far, all known hard digraphs were proved hard by a reduction from RESTRICTED 2-LINKAGE. This paper is no exception: in Section 4, we show that some digraphs are NP-hard with such reductions.

2.4 Known results and tools for F-SUBDIVISION

Let F be a digraph and u a vertex in F. In an F-subdivision S, the vertex corresponding to u is called the *u-vertex of S*. A vertex corresponding to some vertex $u \in F$ is called an *original* vertex.

Bang-Jensen et al. [2] proved that, given a digraph D and a vertex z in D, one can decide in polynomial time if D contains a W_2-subdivision with centre z. Therefore W_2-SUBDIVISION is polynomial-time solvable. We now prove that we can also decide in polynomial time if there is a W_2-subdivision with two prescribed original vertices.

Lemma 11. *Let W_2 be the 2-wheel with centre c and rim aba. Given a digraph D and two vertices b' and c', one can decide in $O(n^2(n+m))$ time if there is a W_2-subdivision in D with b-vertex b' and c-vertex c'.*

Proof. Let us call a W_2-subdivision with b-vertex b' and c-vertex c' a (b', c')*-forced* W_2-subdivision. Let S be the strong component of b' in $D - c'$. The key element is the following claim.

Claim 11.1. *D contains a (b', c')-forced W_2 subdivision if and only if there exist distinct vertices x_1 and x_2 in $V(S)$ such that there are two independent $(c', \{x_1, x_2\})$-dipaths P_1 and P_2 in $D - (S \setminus \{x_1, x_2\})$ and there are two independent $(\{x_1, x_2\}, b')$-dipaths Q_1 and Q_2 in S.*

Subproof. Clearly, existence of two vertices x_1, x_2 and four dipaths P_1, P_2, Q_1, Q_2 as in the statement is a necessary condition for the existence of a (b', c')-forced W_2-subdivision. Let us now prove that it is also sufficient. Assume that such vertices x_1, x_2 and dipaths P_1, P_2, Q_1, Q_2 exist. Since S is strong, it contains a dipath R from b' to $(V(Q_1) \cup V(Q_2)) \setminus \{b'\}$. (This set is not empty since it contains $\{x_1, x_2\} \setminus \{b'\}$.) Then $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R$ is a (b', c')-forced W_2-subdivision. $\lozenge$

Our algorithm is the following. We first compute S, which can be done in linear time. Then for every pair $\{x_1, x_2\}$ of vertices of S, we check by running twice a Menger algorithm if the dipaths P_1 and P_2, and Q_1 and Q_2 as described in Claim 11.1 exist. If yes, we return 'yes', otherwise we return 'no'. The validity of this algorithm is given by Claim 11.1. Since there are $O(n^2)$ pairs of vertices $\{x_1, x_2\}$, the algorithm runs in $O(n^2(n+m))$ time. $\square$

A *spider* is a tree obtained from disjoint directed paths by identifying one end of each path into a single vertex. This vertex is called the *body* of the spider. Observe that if T is a spider, then every T-subdivision contains T as a subdigraph. Hence a digraph contains a T-subdivision if and only if it contains D as a subdigraph. This implies that T-SUBDIVISION can be solved in $O(n^{|T|})$ time. It also easily implies the following.

Lemma 12. *Let F be a digraph and T a spider. If F is tractable, then $F + T$ is also tractable.*

Gluing a spider T with body b to F at a vertex $u \in V(F)$ consists in taking the disjoint union of F and T and identifying u and b.

Lemma 13. *Let F be a digraph and u a vertex of F. If given a digraph D and a vertex v of D, one can decide in polynomial time if there is an F-subdivision in D such that v is the u-vertex, then any digraph obtained from F by gluing a spider at u is tractable.*

Proof. Let T be a spider with body b and let F' be the digraph obtained by gluing T to F at u. Clearly, every F'-subdivision contains an F'-subdivision in which the arcs of T are not subdivided. Such an F'-subdivision is said to be *canonical*.

Consider the following algorithm. For every vertex v we repeat the following. For every set W of $|V(T)| - 1$ vertices, we check whether $D[W \cup \{v\}]$ contains a copy of T with body v. This can be done in constant time. Then we check if $D - W$ contains an F-subdivision with u-vertex v. This can be done in polynomial time by our assumption.

This algorithm clearly decides in polynomial time whether a given digraph D contains a canonical F'-subdivision. $\square$

A $(k_1, \dots, k_p)$-*spindle* is the union of p pairwise internally disjoint (a, b)-dipaths $P_1, \dots, P_p$ of respective lengths $k_1, \dots, k_p$. The vertex a is said to be the *tail* of the spindle and b its *head*. Bang-Jensen et al. [2] proved that spindles are tractable. Their proof uses the following result.

Theorem 14 (Bang-Jensen et al. [2])**.** *Let F be a spindle with tail a and head b. Given a digraph D and two vertices a' and b', we can decide in polynomial time if T contains an F-subdivision with a-vertex a' and b-vertex b'.*

The $(k_1, \dots, k_p; l_1, \dots, l_q)$-*bispindle*, denoted $B(k_1, \dots, k_p; l_1, \dots, l_q)$, is the digraph obtained from the disjoint union of a $(k_1, \dots, k_p)$-spindle with tail a_1 and head b_1 and an $(l_1, \dots, l_q)$-spindle with tail a_2 and head b_2 by identifying a_1 with b_2 into a vertex a, and a_2 with b_1 into a vertex b. The vertices a and b are called, respectively, the *left node* and the *right node* of the bispindle. The directed (a, b)-paths are called the *forward paths*, while the directed (b, a)-paths are called the *backward paths*. Bang-Jensen et al. [2] proved that a bispindle is hard if and only if $p \geq 1$, $q \geq 1$ and $p + q \geq 4$. To prove that a bispindle with two forward paths and one backward path is tractable, they provided the following theorem.

Theorem 15 (Bang-Jensen et al. [2])**.** *Let F be a bispindle with two forward paths and one backward path, and let x be one of its nodes. Given a digraph D and a vertex a', we can decide in polynomial time if D contains an F-subdivision with a-vertex a'.*

Lemma 16. *Let F be a digraph and let $u_1, \dots, u_p$ be distinct vertices of F. Suppose that for every outneighbour v of u_1, replacing the arc u_1v by a dipath u_1wv of length 2, where $w \notin V(F)$, always results in the same digraph F'. Suppose that for every given digraph D of order n and p vertices $x_1, \dots, x_p$ in D, one can decide in $f(n)$ time whether there is an F-subdivision in D such that x_i is the u_i-vertex for every i. Then given a digraph D and p vertices $x_1, \dots, x_p$, one can decide in $O\left(\binom{d^+(x_1)-1}{d^+(u_1)-1} \sum_{y \in N^+(x_1)} d^+(y) \cdot f(n-1)\right)$ time whether there is an F'-subdivision in D such that x_i is the u_i-vertex for every i.*

Proof. Set $q = d^+(u_1)$. For every set of q neighbours $y_1, \dots, y_q$ of x_1 and every outneighbour z of y_1, where $z \notin \{y_2, \dots, y_q\}$, we shall give a procedure that verifies if D contains an F'-subdivision S' such that x_i is the u_i-vertex for all $1 \le i \le p$, and $\{x_1y_1, \dots, x_1y_q, y_1z\} \subseteq A(S')$. Such an F'-subdivision is called *forced*.

Let D' be the digraph obtained from $D - y_1$ by deleting all arcs leaving x_1 except $x_1y_2, \dots, x_1y_q$, and adding the arc x_1z.

Claim 16.1. *D has a forced F'-subdivision if and only if D' has an F-subdivision such that x_i is the u_i-vertex for every i.*

Subproof. Suppose that S is an F-subdivision in D' such that x_i is the u_i-vertex for all i. Since x_1 has outdegree q in D', we have $\{x_1y_2, \dots, x_1y_q, x_1z\} \subseteq A(S)$. Let S' be the digraph obtained from S by replacing the arc x_1z by the dipath x_1y_1z. Because replacing the arc u_1v by a dipath of length 2 results in F' for any outneighbour v of u_1, the digraph S' is an F'-subdivision in D. Thus S' is a forced F'-subdivision in D.

Conversely, assume that S' is a forced F'-subdivision in D. Then the digraph S obtained from S' by replacing the dipath x_1y_1z by the arc x_1z is an F-subdivision in D' such that x_i is the u_i-vertex for every i. $\Diamond$

This claim implies that deciding whether D contains a forced F'-subdivision can be done by checking whether D' has an F-subdivision such that x_i is the u_i-vertex for all i. This can be done in $f(n-1)$ time by assumption. By repeating this for every possible set $\{y_1, \dots, y_q, z\}$ where the y_i are distinct outneighbours of x_1 and $z \notin \{y_2, \dots, y_q\}$ is an outneighbour of y_1, we obtain an algorithm to decide whether there is an F'-subdivision in D such that x_i is the u_i-vertex for all i. Since there are at most $\binom{d^+(x_1)-1}{d^+(u_1)-1} \sum_{y \in N^+(x_1)} d^+(y)$ such sets, the running time of this algorithm is as claimed. $\square$

3 Oriented graphs of order 4

The aim of this section is to prove that every oriented graph of order 4 is tractable.

Theorem 17. *If F is an oriented graph of order 4, then F-SUBDIVISION is polynomial-time solvable.*

Proof. If F has no big vertices, then by Theorem 2, F-SUBDIVISION is polynomial-time solvable. Henceforth, we assume that F has at least one big vertex. Free to consider its converse, we may assume that F has a vertex with out-degree 3.

If F is a tournament, then it is either the transitive tournament TT_4. Bang-Jensen et al. [2] (Theorem 64) proved that TT_4-SUBDIVISION is polynomial-time solvable. We show in Subsection 3.1 that W_3 is tractable.

If F is an orientation of $K_4 \setminus e$, the graph obtained from K_4 by removing one edge, then F must be one of the oriented graphs depicted Figure 4, or the converse of one of those. $S(1,2,2)$ is a spindle and F_3 is the 3-fan. These digraphs have been shown tractable in [2] (Proposition 20 and Theorem 61). We prove in Subsection 3.2 that TT_4' is tractable, and in Subsection 3.3 that Z_4 is tractable.

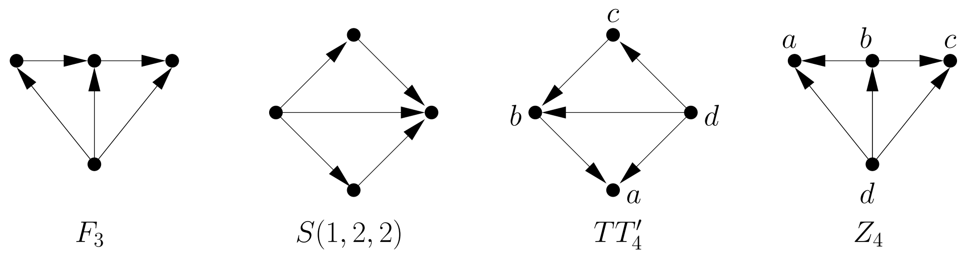


Figure 4: Some orientations of $K_4 \setminus e$

If D has at most four arcs, then it is either a star or a star plus an arc. Those digraphs have been proved tractable in [2]. □

3.1 Subdivision of the 3-wheel

Theorem 18. W_3-SUBDIVISION *can be solved in* $O(n^6(n+m))$ *time.*

The proof of this theorem relies on the following notion. Let X be a set of three vertices. An X-*tripod* is a digraph which is the union of a directed cycle C and three disjoint dipaths P_1, P_2, P_3 with initial vertices in X and terminal vertices in C. If the P_i are (X, C)-dipaths, we say that the tripod is *unfolded*. Note that the dipaths P_i may be of length 0. We shall denote the tripod described above as the 4-tuple (C, P_1, P_2, P_3).

Proposition 19. *Let* $X = \{x_1, x_2, x_3\}$ *be a set of three distinct vertices. Any* X*-tripod contains an unfolded* X*-tripod.*

We shall consider the following decision problem.

TRIPOD
Input: A strong digraph D and a set X of three distinct vertices of D.
Question: Does D contain an X-tripod?

Lemma 20. TRIPOD *can be solved in* $O(n^2(n+m))$ *time.*

Proof. Let us describe a procedure `tripod`(D, X), solving TRIPOD.

We first look for a directed cycle of length at least 3 in D. This can be done in linear time. If there is no such cycle, then we return 'no'.

Otherwise we have a directed cycle C of length at least 3. We choose a set Y of three vertices in C and run a Menger algorithm between X and Y. If such an algorithm finds three disjoint (X, Y)-dipaths P_1, P_2, P_3, then we return the tripod (C, P_1, P_2, P_3). Otherwise, the Menger algorithm finds a 2-separation (W, S, Z) of (X, Y). Note that $|S| \geq 1$ because D is strong.

Assume first that $|S| = 1$, say $S = \{s\}$. Let D_1 be the digraph obtained from $D[W \cup S]$ by adding the arc sw for every vertex w in W having an inneighbour $z \in Z$. We then make a recursive call to `tripod`(D_1, X). This is valid by virtue of the following claim.

Claim 20.1. *There is an* X*-tripod in* D *if and only if there is an* X*-tripod in* D_1.

Subproof. Suppose first that there is an X-tripod in D_1. Then D_1 contains an unfolded X-tripod T_1 by Proposition 19. If T_1 is contained in D, then we are done. So we may assume that it is not. Then T_1 contains an arc $sw \in A(D_1) \setminus A(D)$. It can contain only one such arc since every vertex has outdegree at most one in T_1 and all such arcs leave s. Furthermore, the head w of this arc is in W and w has an inneighbour z in Z. Now, since D is strong, there is an (s, z)-dipath Q in D. Because there is no arc from W to Z, all internal vertices of Q are in Z. Hence the digraph T obtained from T_1 by replacing the arc sw by the dipath Qzw is an X-tripod in D.

Suppose now that D contains an X-tripod. Then it contains an unfolded X-tripod $T = (C, P_1, P_2, P_3)$ by Proposition 19. Since all (X, Z)-dipaths in D go through s, the terminal vertices of the P_i are in $W \cup S$, and $D[Z] \cap T$ is a dipath Q which is a subpath of one of the P_i or C. If Q is a (t, z)-dipath, then T contains arcs st and zw for some $w \in W$. Then the digraph T_1 obtained from T by replacing sQw by the arc sw is an X-tripod in D_1. $\diamondsuit$

Assume now that $|S| = 2$, say $S = \{s_1, s_2\}$. If there is no arc from Z to W, let D_2 be the digraph obtained from $D[W \cup S]$ by adding the arc s_1s_2 (resp. s_2s_1) (if the arc is not already present in D) if there is an (s_1, s_2)-dipath (resp. (s_2, s_1)-dipath) in $D[Z \cup S]$. We then make a recursive call to $\texttt{tripod}(D_2, X)$. This is valid by virtue of the following claim.

Claim 20.2. *There is an X-tripod in D if and only if there is an X-tripod in D_2.*

Subproof. Suppose first that there exists an X-tripod in D_2. Then there is an unfolded X-tripod T_2 in D_2, by Proposition 19. Then either it is an X-tripod in D, or T_2 contains exactly one of the arcs s_1s_2, s_2s_1 and this arc is not in $A(D)$. Without loss of generality, we may assume that this arc is s_1s_2. Since $s_1s_2 \in A(D_2) \setminus A(D)$, there is an (s_1, s_2)-dipath Q in $D[Z \cup S]$. Hence the digraph T obtained from T_2 by replacing the arc s_1s_2 by the dipath Q is an X-tripod in D.

Suppose now that D contains an X-tripod. Then it contains an unfolded X-tripod $T = (C, P_1, P_2, P_3)$ by Proposition 19. For $i = 1, 2, 3$, let y_i be the terminal vertex of P_i. Without loss of generality, we may assume that y_1, y_2, y_3 appear in this order along C. Since all $(X, Z \cup S)$-dipaths intersect S, one of the y_i, say y_3, must be in W. The three oriented paths P_2, $P_1C[y_1, y_2]$, and $\overline{C}[y_3y_2]$ are independent (W, y_2)-paths. But the graph underlying D has no edges between W and Z, by the assumption made in the current subcase. So y_2 is in $W \cup S$. Similarly, y_1 is in $W \cup S$. It follows that $T \cap D[Z]$ is a dipath Q which is a subpath of one of the P_i or C. Moreover, the inneighbour in T of the initial vertex of Q is some vertex $s \in S$ (because there is no arc from W to Z) and the outneighbour in T of the terminal vertex of Q is some vertex $s' \in S$ because there is no arc from Z to W). Furthermore $s \neq s'$ for otherwise $sQs' = C$ which is impossible as since $y_3 \in W \cap C$. Moreover, because sQs' is an (s, s')-dipath in $D[Z \cup S]$, ss' is an arc in D_2. Thus the digraph T_2 obtained from T by replacing sQs' by the arc ss' is an X-tripod in D_2. $\diamondsuit$

Now we may assume that there is an arc z_1w_1 with $z_1 \in Z$ and $w_1 \in W$. Since D is strong, there is a cycle C' containing the arc z_1w_1. Necessarily, the cycle C' must go through S and it contains at least three vertices.

<u>Case 1</u>: $S \subset V(C')$. Set $Y' = \{w_1, s_1, s_2\}$. We run a Menger algorithm between X and Y'. If such an algorithm finds three disjoint (X, Y')-dipaths P'_1, P'_2, P'_3, then we return the X-tripod (C', P'_1, P'_2, P'_3).

If not, we obtain a 2-separation (W', S', Z') of (X, Y'). We claim that $|W'| < |W|$. Indeed, no vertex $z \in Z$ is in W' because every (X, z)-dipath must go through S and thus through S'. Hence $W' \subseteq W \setminus \{w_1\}$. Now, we replace C by C', Y by Y' and (W, S, Z) by (W', S', Z'), and then redo the procedure.

<u>Case 2</u>: $|S \cap V(C')| = 1$. Without loss of generality, we may assume $S \cap V(C') = \{s_1\}$. Set $Y' = \{w_1, s_1, z_1\}$. As in Case 1, we run a Menger algorithm between X and Y'. If such an algorithm finds three disjoint (X, Y')-dipaths P'_1, P'_2, P'_3, then we return the X-tripod (C', P'_1, P'_2, P'_3).

If not, the Menger algorithm returns a 2-separation (W', S', Z') for (X, Y'). Observe that there is a vertex $s'_1 \in S' \cap W$ because w_1 is reachable from X in $D[W]$. If S' contains a vertex s'_2 in Z, then one can see that there are no (X, Y')-dipaths in $D - \{s'_1, s_2\}$. Thus, there is a 2-separation (W'', S'', Z'') of (X, Y') where $S'' \subseteq \{s'_1, s_2\}$ and $s_1 \in Z''$. Hence, after possibly replacing the 2-separation (W', S', Z') by (W'', S'', Z''), we may assume that $S' \subset W \cup S$.

If $|W'| < |W|$, then we set $C := C'$, $Y := Y'$, $(W, S, Z) := (W', S', Z')$, and redo the procedure.

If not, then the set $R = Z \cap W'$ is not empty. Set $L = Z \setminus R = Z \cap Z'$. There is no arc from R to L, because (W', S', Z') is a 2-separation. Moreover, all (X, R)-dipaths must go through s_2. In particular, $s_2 \in W'$. Let D_3 be the digraph obtained from $D - L$ by adding an arc s_1w for every $w \in W$ having an inneighbour in L. We then make a recursive call to $\texttt{tripod}(D_3, X)$. This is valid by virtue of the following claim.

Claim 20.3. *There is an X-tripod in D if and only if there is an X-tripod in D_3.*

Subproof. Suppose first that D_3 contains an X-tripod. Then it contains an unfolded X-tripod T_3 by Proposition 19. If T_3 is contained in D, then we are done. So we may assume that T_3 is not contained in D. Then T_3 contains an arc in $s_1w \in A(D_3) \setminus A(D)$. It contains only one such arc since every vertex has outdegree at most one in T_3 and all arcs of $A(D_3) \setminus A(D)$ leave s_1. Furthermore the head w of this arc is in W and has an inneighbour $z \in L$. Since D is strong, there is an (s_1, z)-dipath Q in D. Moreover since $s_2 \in W'$ all the (s_2, z)-dipaths must go through S'. But $S' \subseteq W \cup \{s_1\}$, so all (s_2, z)-dipaths must go through s_1. Thus Q does not go through s_2. It follows that all internal vertices of Q are in Z, because (W, S, Z) is a 2-separation, and so in L because there is no arc from R to L. Consequently, the digraph T obtained from T_3 by replacing the arc s_1w by the dipath Qzw is an X-tripod in D.

Suppose now that D contains an X-tripod. Then it contains an unfolded X-tripod $T = (C, P_1, P_2, P_3)$ by Proposition 19. For $i = 1, 2, 3$, let y_i be the terminal vertex of P_i. Without loss of generality, we may assume that y_1, y_2, y_3 appear in this order along C. If T is contained in $D - L$, then it is an X-tripod in D_3. Hence we may assume that T contains some vertices of L. Observe that the arcs entering L all leave s_1. Hence, y_i cannot be in L, since there are two (X, y_i)-dipaths in T, which are disjoint except for the common vertex y_i. Consequently, the intersection of T with $D[L]$ is a dipath Q which is a subpath of one of the P_i or C. Moreover, the inneighbour in T of the initial vertex of Q is s_1 and the outneighbour in T of the terminal vertex of Q is some vertex $w \in W \cup \{s_1\}$, because there is no arc from L to $R \cup \{s_2\}$. But $w \neq s_1$ for otherwise s_1Qs_1 would be C and would contain at most one of the y_i, a contradiction. Thus the digraph T_3 obtained from T by replacing s_1Qw by the arc s_1w is an X-tripod in D_3. $\lozenge$

Claims 20.1, 20.2 and 20.3 ensure that our algorithm is correct. Each time we do a recursive call, the number of vertices decreases. So we do at most n of them. Between two recursive calls, we first find a cycle of length at least 3 in linear time, and next run a sequence of Menger algorithms to produce a new 2-separation. At each step the size of the set W decreases. Therefore, we run at most n times the Menger algorithm between two recursive calls. Since a Menger algorithm runs in linear time, the time between two calls is at most $O(n(n + m))$ and so $\texttt{tripod}$ runs in $O(n^2(n + m)$ time. $\square$

With Lemma 20 in hands, we now deduce Theorem 18.

Proof of Theorem 18. For every vertex v, we examine whether there is a W_3-subdivision with centre v in D. Observe that such a subdivision S is the union of a directed cycle C, and three internally disjoint (v, C)-dipaths P_1, P_2, P_3 with distinct terminal vertices y_1, y_2, y_3. The cycle C is contained in some strong component Γ of $D - v$. For $i = 1, 2, 3$, let x_i be the first vertex of P_i that belongs to Γ. Set $X = \{x_1, x_2, x_3\}$. Then the paths $P_i[x_i, y_i]$, $i = 1, 2, 3$, and C form an X-tripod in Γ, and the $P_i[v, x_i]$, $i = 1, 2, 3$, are internally disjoint (v, X)-dipaths in $D - (\Gamma \setminus X)$.

Hence for finding a W_3-subdivision with centre v, we use the following procedure to check whether there is a set X as above. First, we compute the strong components of $D - v$. Next, for every subset X of three vertices in the same strong component Γ, we run a Menger algorithm to check whether there are three independent (v, X)-dipaths in $D - (\Gamma \setminus X)$. If yes, we check whether there is an X-tripod in Γ. If yes again, then we clearly have a W_3-subdivision with centre v, and we return 'yes'. If not, there is no such subdivision, and we proceed to the next triple.

For each vertex v, there are at most n^3 possible triples. And for each triple we run a Menger algorithm in time $O(n+m)$ and possibly `tripod` in time $O(n^2(n+m))$. Hence the time spent on each vertex v is $O(n^5(n+m))$. As we examine at most n vertices, the algorithm runs in $O(n^6(n+m))$ time. □

3.2 TT_4'-subdivision

In this subsection, we prove that TT_4' is tractable. Our proof relies on the notion of *good triple*. A triple of distinct vertices (a', b', d') is *good* if there are a (b', a')-dipath Q in $D - d'$ and three internally disjoint dipaths P_1, P_2, P_3 with $s(P_1) = s(P_2) = s(P_3) = d'$, $t(P_1) = t(P_2) = b'$, and $t(P_3) = a'$.

Proposition 21. *A digraph D contains a TT_4'-subdivision if and only if it has a good triple.*

Proof. If D contains a TT_4'-subdivision, then the triple formed by its a-vertex, its b-vertex and its d-vertex is good.

Conversely, suppose that D contains a good triple. Let (a', b', d') be a good triple that minimizes the sum of the lengths of the paths Q, P_1, P_2, P_3 as named in the definition.

Assume for a contradiction that Q° intersects P_3°. Let a'' be a vertex of $Q^\circ \cap P_3^\circ$. Then the triple (a'', b', d') is good because of the paths $Q[b', a'']$, P_1, P_2, $P_3[d', a'']$, and contradicts the minimality of (a', b', d'). Hence Q° does not intersect P_3°.

Assume for a contradiction that Q° intersects $P_1^\circ \cup P_2^\circ$. Let b'' be the first vertex along Q° in $P_1^\circ \cup P_2^\circ$. Without loss of generality, we may assume that b'' is on P_1°. Then the triple (a', b'', d') is good because of the paths $Q[b'', a']$, $P_1[d', b'']$, $Q[b', b'']P_2$, P_3, and contradicts the minimality of (a', b', d'). Hence Q° does not intersect $P_1^\circ \cup P_2^\circ$.

Therefore the paths Q, P_1, P_2, P_3 are internally disjoint and the union of those dipaths is a TT_4'-subdivision. □

Corollary 22. TT_4'-SUBDIVISION *can be solved in $O(n^5(n+m))$ time.*

Proof. According to Proposition 21, TT_4'-SUBDIVISION is equivalent to deciding if D has a good triple.

Now one can decide if a triple (a', b', d')-triple in $O(n^2(n+m)$ time as follows. We check if there is an (a', b')-dipath Q in $D - d'$, and for every pair s_1, s_2 of distinct outneighbours of b' in $D - a'$, we check if there are three independent $(\{s_1, s_2, a\}, d')$-dipath in $D - b'$ by a Menger algorithm.

Doing this procedure for the $O(n^3)$ triple of distinct vertices of D, one decides in $O(n^5(n+m)$ time whether D has a good triple. □

3.3 Z_4-subdivision

In this subsection, we show that Z_4 is tractable. The proof relies on the following lemma.

Lemma 23. *Let D be a digraph. There is a Z_4-subdivision in D if and only if there exists four distinct vertices a', b', c' and d' in D such that the following hold.*

(i) There are three independent $(d', \{a', b', c'\})$-dipaths.

(ii) There are two independent $(b', \{a', c'\})$-dipaths.

Proof. If D contains a Z_4-subdivision S, then the vertices a', b', c', d' corresponding to a, b, c, d (as indicated on Figure 4) clearly satisfy conditions (i) and (ii).

Conversely, suppose that D contains four vertices a', b', c', d' satisfying conditions (i) and (ii). Let P_1, P_2, P_3 be three independent $(d', \{a', b', c'\})$-dipaths with $t(P_1) = a'$, $t(P_2) = b'$ and $t(P_3) = c'$; let Q_1, Q_2 be two independent $(b', \{a', c'\})$-dipaths with $t(Q_1) = a'$ and $t(Q_2) = c'$.

We consider such vertices a', b', c', d' and dipaths such that the sum of the lengths of P_1, P_2, P_3, Q_1 and Q_2 is minimized.

Claim 23.1. *$V(Q_1) \cap V(P_1) = \{a'\}$ and $V(Q_2) \cap V(P_3) = \{c'\}$.*

Subproof. Suppose $V(Q_1) \cap V(P_1) \neq \{a'\}$. Then there is a vertex a'' distinct from a' in $V(Q_1) \cap V(P_1)$. The vertices a'', b', c', d' satisfy condition (i) with $P_1[d', a'']$, P_2, P_3 and condition (ii) with $Q_1[b', a'']$, Q_2. This contradicts our choice of a', b', c', d' and the corresponding paths, and so $V(Q_1) \cap V(P_1) = \{a'\}$.

The conclusion that $V(Q_2) \cap V(P_3) = \{c'\}$ is proved in the same way; the details are omitted. $\Diamond$

Claim 23.2. $(V(Q_1) \cup V(Q_2)) \cap V(P_2) = \{b'\}$.

Subproof. Suppose not. Then let b'' be the last vertex distinct from b' along P_2 which is in $V(Q_1) \cup V(Q_2)$. By symmetry, we may assume that $b'' \in V(Q_1)$. But the four vertices a', b'', c', d' satisfy condition (i) with P_1, $P_2[d', b'']$, P_3 and condition (ii) with $Q_1[b'', a']$, $P_2[b'', b']Q_2$. This contradicts our choice of a', b', c', d' and proves our claim. $\Diamond$

Claim 23.3. *$V(Q_1) \cap V(P_3) = \emptyset$ and $V(Q_2) \cap V(P_1) = \emptyset$.*

Subproof. Suppose not. Then $V(Q_1) \cap V(P_3)$ or $V(Q_2) \cap V(P_1)$ is not empty.

Assume first that these two sets are both non-empty. Let a'' be a vertex in $V(Q_2) \cap V(P_1)$ and c'' be a vertex in $V(Q_1) \cap V(P_3)$. Then the four vertices a'', b', c'', d' satisfy condition (i) with $P_1[d', a'']$, $P_3[d', c'']$, P_2 and condition (ii) with $Q_2[b', a'']$, $Q_1[b', c'']$. This contradicts our choice of a', b', c', d'.

Hence, exactly one of the two sets is empty. By symmetry, we may assume that $V(Q_1) \cap V(P_3) \neq \emptyset$, Let b'' be a vertex in $V(Q_1) \cap V(P_3)$. Now the four vertices a', b'', c', d' satisfy condition (i) with P_1, $P_3[d', b'']$, P_2Q_2 and condition (ii) with $Q_1[b'', a']$, $P_3[b'', c']$. This contradicts our choice of a', b', c', d' and proves our claim. $\Diamond$

Claims 23.1, 23.2 and 23.3 imply that $P_1 \cup P_2 \cup P_3 \cup Q_1 \cup Q_2$ is a Z_4-subdivision. $\square$

Theorem 24. *Z_4-SUBDIVISION can be solved in $O(n^4(n+m))$ time.*

Proof. By Lemma 23, Z_4-SUBDIVISION is equivalent to deciding whether there are four vertices satisfying the condition (i) and (ii) of the lemma. But given four vertices a', b', c', d', one can check in linear time if conditions (i) and (ii) hold by running two Menger algorithms. Since there are $O(n^4)$ sets of four vertices in D, Z_4-SUBDIVISION can be solved in $O(n^4(n+m))$ time. $\square$

4 Some hard digraphs

Theorem 9 and more specifically Corollary 10 imply that many digraphs on 4 vertices are hard. We now prove that some additional digraphs that are not covered by Theorem 9 are also hard. These graphs are depicted in Figure 5, where each of the bold edges without indicated direction represents a pair of oppositely directed arcs.

Proposition 25. *For each digraph N_i, $1 \leq i \leq 9$, depicted Figure 5, N_i-SUBDIVISION is NP-complete.*

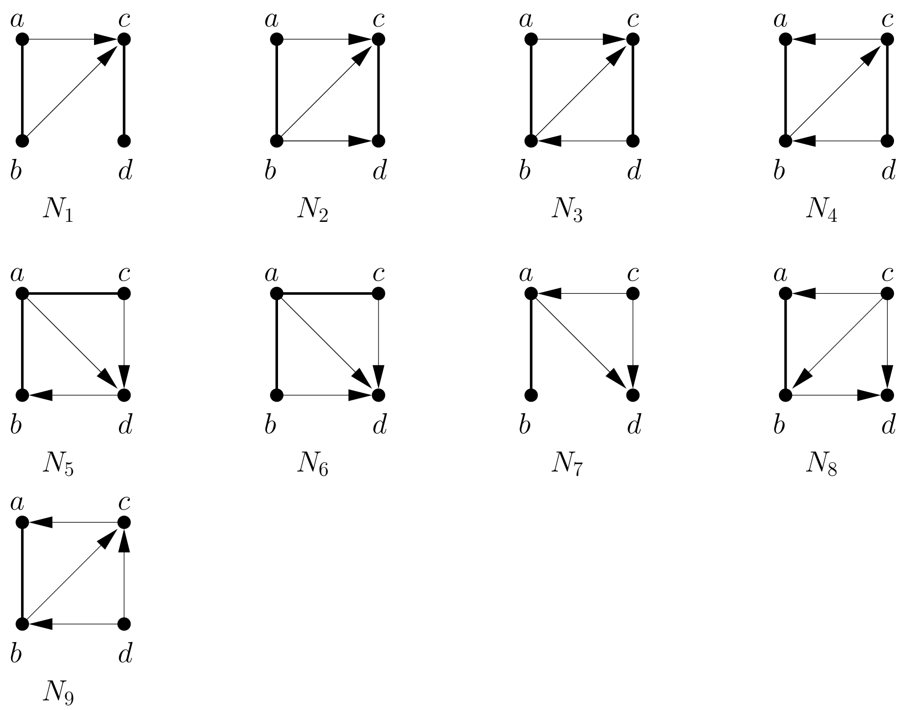


Figure 5: Some hard digraphs on 4-vertices. Bold undirected edges represent directed 2-cycles.

Proof. In each case, the problem is proved to be NP-complete by reduction from RESTRICTED 2-LINKAGE. Let D, x_1, x_2, y_1 and y_2 be an instance of this problem. We construct a digraph D_i by *putting* D on two arcs $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of N_i (that will be specified later), that is by taking the disjoint union of D and N_i, by removing the arcs e_1 and e_2 and adding the arcs u_1x_1, y_1v_1, u_2x_2 and y_2v_2. We then show that D_i contains an N_i-subdivision if and only if there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D. This implies that N_i-SUBDIVISION is NP-complete.

Clearly, by construction of D_i, if there is a 2-linkage from (x_1, x_2) to (y_1, y_2) in D, then D_i contains an N_i-subdivision. We now prove the converse for each i. In each case we shall assume that D_i contains an N_i-subdivision S, and we shall denote by a', b', c', d' the vertices in S corresponding to a, b, c, d, respectively.

$i = 1$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $c' = c$. Because $d^-_{D_1}(c) = 3$, the arcs ac, bc and dc are in S. Moreover, the arc ba is in S, because every vertex has indegree at least 1 in S. Thus $d^+_S(b) \geq 2$, and so either $b = b'$ or $b = a'$. By symmetry between a and b in N_1, we may assume that $b = b'$. Then, necessarily, $a = a'$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i \in \{2, 3, 4\}$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $\{b, c\} = \{b', c'\}$. Therefore, the arc bc is contained in S, and this shows that $b' = b$ and $c' = c$. Now for degree reasons, all arcs incident to b and c must be in S. It follows that $a' = a$ and $d' = d$. (This is clear for N_3 and N_4. For N_2, we first conclude that $\{a', d'\} = \{a, d\}$ and then consider degrees of a and d to obtain the same conclusion.) Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induced a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 5$: We choose $e_1 = ba$ and $e_2 = cd$. Since D contains no big vertices, we have $a' = a$. Hence all the arcs incident to a are in $A(S)$. Therefore c is ether b' or c'. But $d^-(c) = 1$, so c cannot be b', and thus $c = c'$. All vertices have outdegree at least 1 in S, so $db \in A(S)$. Now there are two internally disjoint (a', b)-dipaths in $S - c'$, so necessarily, $b = b'$. Moreover, d' must be in one of those dipaths, so $d = d'$. Therefore, in S, there are internally disjoint (b, a)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 6$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $a' = a$ and $d' = d$. Hence all arcs incident to those two vertices are in S. Therefore $\{b', c'\} = \{b, c\}$. By symmetry of N_6, we may assume that $b' = b$ and $c' = c$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 7$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $a' = a$. Hence all arcs incident to a are in S. So c and d are in $V(S)$. Since $d^+_{D_7}(d) = 0$, we have $d = d'$; since $d^-_{D_7}(c) = 0$, we have $c = c'$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 8$: We choose $e_1 = ab$ and $e_2 = cd$. Since D contains no big vertices, we have $b' = b$ and $c' = c$. Hence all arcs incident to those two vertices are in S. So $d \in V(S)$. Since $d^+_{D_8}(d) = 0$, it follows that $d = d'$. The arcs ba and ca show that $d^-_S(a) \geq 2$. Thus $a = a'$. Therefore, in S, there are disjoint (a, b)- and (c, d)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D.

$i = 9$: We choose $e_1 = ab$ and $e_2 = dc$. Since D contains no big vertices, we have $b' = b$. Hence all arcs incident to b are in S. In particular $c, d \in V(S)$. Since $d^-_{D_9}(d) = 0$, we have $d' = d$. Since $d^+_S(c) \geq 1$, the arc ca is in $A(S)$, so $d^-_S(a) = 2$, and thus $a \in \{a', c'\}$. Since a' and c' are both in the outsection of d in $N_9 - b$, S contains a (d, a)-dipath disjoint from b. This dipath must pass through c and therefore the arc y_2c lies in S. This implies that $d^-_S(c) \geq 2$, so $c = c'$ and then we have $a = a'$. Consequently, in S, there are disjoint (a, b)- and (d, c)-dipaths. These two paths induce a 2-linkage from (x_1, x_2) to (y_1, y_2) in D. □

5 Some tractable digraphs

5.1 Easier cases

A *symmetric star* is a symmetric digraph associated to a star. The *centre* of a symmetric star is the centre of the star to which it is associated. A *superstar* is a digraph obtained from a symmetric star by adding an arc joining two non-central vertices. The *centre* of a superstar is the centre of the star from which it is derived. The symmetric star of order $k+1$ is denoted by SS_k and the superstar of order $k+1$ is denoted by SS^*_k. An SS_k-subdivision with centre a is the union of k internally disjoint (a, a)-handle. Therefore, on can decide if there is an SS_k-subdivision with centre a in linear time using a Menger algorithm. Bang-Jensen et al. [2] showed that SS^*_3-SUBDIVISION is polynomial-time solvable. Their result can be extended to all superstars.

Theorem 26. *Let k be a positive integer. Given digraph D and a vertex v of D, on can decide in $O(n^{2k}(n+m))$-time whether D contains an SS^*_k-subdivision with centre v.*

Proof. We describe a procedure that given v, a set $X = \{x_1, \dots, x_k\}$ of k distinct outneighbours of v and a set $Y = \{y_1, \dots, y_k\}$ of k distinct inneighbours of v checks if there is an SS^*_k-subdivision S with centre v such that $\{vx_1, \dots, vx_k\} \cup \{y_1v, \dots, y_kv\} \in A(S)$. (Observe that it is allowed that $X \cap Y \neq \emptyset$.) Such a subdivision will be called (v, X, Y)*-forced*.

Applying a Menger algorithm, check whether in $D - v$ there are k disjoint dipaths $P_1, \dots, P_k$ from X to Y. If not, then D certainly does not contain any (v, X, Y)-forced SS_k-subdivision. If yes, then check whether there is a dipath Q from some P_i to a different P_j whose internal vertices are not in

$\{v\} \cup \bigcup_{i=1}^{k} P_i$. This can be done in linear time by running a search on the digraph obtained from $D - v$ by contracting each path P_i into a single vertex. If such a Q exists, then $P_1, \dots, P_k$ and Q together with v and the arcs from v to X and from Y to v form a (v, X, Y)-forced SS_k^*-subdivision. If not, then no (v, X, Y)-forced SS_k^*-subdivision using the chosen arcs exists, because there is no vertex $x \in X$ with two vertices of Y in its outsection in $D - v$.

Applying this linear-time procedure, for every possible pair (X, Y), we can decide in $O(n^{2k}(n+m))$-time whether D contains an SS_k^*-subdivision with centre v. □

Corollary 27. *For every positive integer k, SS_k^*-*SUBDIVISION *can be solved in $O(n^{2k+1}(n+m))$-time.*

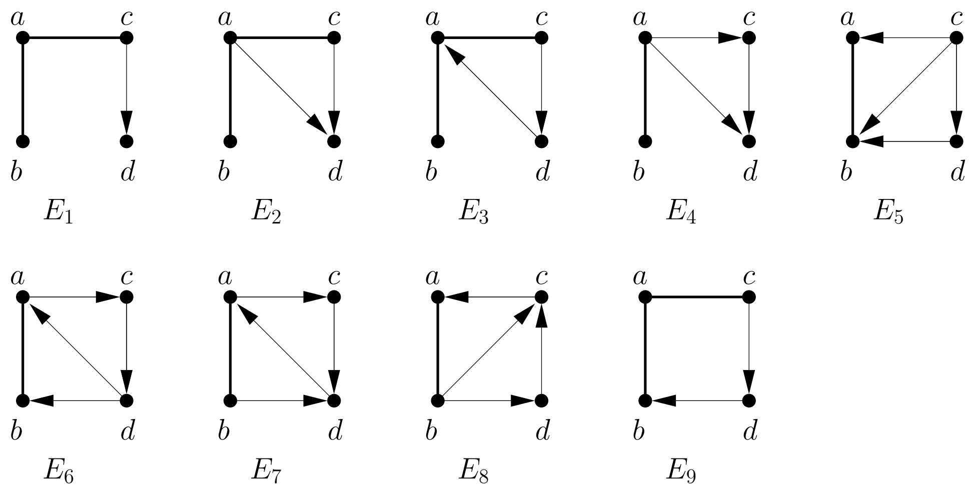


Figure 6: Some digraphs on 4-vertices, that are tractable. Bold undirected edges represent directed 2-cycles.

Proposition 28. *For $1 \le i \le 8$, the digraph E_i depicted in Figure 6 is tractable.*

Proof. $i = 1$: Let us describe a procedure that, given two distinct vertices a' and d' in D and two outneighbours s_1, s_2 of a' distinct from d', decides whether there is an E_1-subdivision with a-vertex a' and d-vertex d' such that $a's_1$ and $a's_2$ are arcs of S. Such a subdivision is said to be $(a's_1, a's_2, d)$-*forced*.

We check whether there is a dipath Q from $\{s_1, s_2\}$ to d' in $D - a'$, and with a Menger algorithm we check whether there are two independent $(\{s_1, s_2\}, a')$-dipaths P_1 and P_2 in $D - d'$. If these three dipaths do not exist, then D contains no $(a's_1, a's_2, d)$-forced E_1-subdivision, and we return 'no'. If the three paths Q, P_1, P_2 exist, then we return 'yes'. Indeed, denoting by c' the last vertex along Q in $P_1 \cup P_2$, the digraph $a's_1 \cup P_1 \cup a's_2 \cup P_2 \cup Q[c', d']$ is an $(a's_1, a's_2, d)$-forced E_1-subdivision.

Applying the above procedure for all possible triples $(a's_1, a's_2, d')$, one solves E_1-SUBDIVISION in $O(n^4(n + m))$ time.

$i = 2$: Let us describe a procedure that given two distinct vertices a' and d' in D, a set $U = \{u_1, u_2, u_3\}$ of three outneighbours of a', returns 'yes' if it finds an E_2-subdivision and returns 'no' only if there is no E_2-subdivision with a-vertex a' and d-vertex d' such that $\{a'u_1, a'u_2, a'u_3\} \subseteq A(S)$. Such a subdivision is said to be (a', d', U)-*forced*.

We check with a Menger algorithm whether $|S^-_{D-a'}(d') \cap U| \geq 2$ and whether there are three internally disjoint dipaths P_1, P_2, P_3 with distinct initial vertices in U and with $t(P_1) = t(P_2) = a'$ and $t(P3) = d'$. If these two conditions are not both fulfilled, then D contains no (a', d', U)-forced E_2-subdivision, and we return 'no'. If these conditions are fulfilled, then we return 'yes'. Indeed consider three dipaths P_1, P_2, P_3 as above. Without loss of generality, $s(P_i) = u_i$ for $1 \leq i \leq 3$. Since $|S^-_{D-a'}(d') \cap U| \geq 2$, there exists a $(P_1 \cup P_2, P_3)$-dipath in $D - a'$. Let us denote its terminal vertex by d''. Then the union of the directed cycles $a'u_1P_1$, $a'u_2P_2$, and the dipaths $a'u_3P_3[u_3, d'']$, and Q is an E_2-subdivision.

Applying the above procedure for all possible triples (a', d', U), one solves E_2-SUBDIVISION in $O(n^5(n + m))$ time.

$i = 3$: Let us describe a procedure that given two distinct vertices a' and d' in D and two outneighbours s_1, s_2 of a' distinct from d', returns 'yes' when it finds an E_3-subdivision and returns 'no' only if there is no E_3-subdivision with a-vertex a' and d-vertex d' such that $\{a's_1, a's_2\} \subseteq S$. Such a subdivision is said to be $(a's_1, a's_2, d')$-*forced*.

We check whether there is an $(\{s_1, s_2\}, d')$ dipath Q in $D-a'$ and whether there are three independent $(\{s_1, s_2, d'\}, a')$-dipaths P_1, P_2, P_3 in D. If these two conditions are not both fulfilled, then D contains no $(a's_1, a's_2, d')$-forced E_3-subdivision, and we return 'no'. If these conditions are fulfilled then we return 'yes'.

Indeed, suppose there are four such dipaths Q, P_1, P_2, P_3. We may assume without loss of generality that $s(P_3) = d'$. Denote by c' the last vertex along Q in $P_1 \cup P_2$, and by d'' the first vertex in $Q[c', d']$ which is on P_3. Then the union of the two directed cycles $a's_1P_1a', a's_2P_2a'$ and the dipaths $Q[c', d'']$ and $P_3[d'', a']$ is an E_3-subdivision.

Applying the above procedure for all possible triples $(a's_1, a's_2, d')$, one solves E_3-SUBDIVISION in $O(n^4(n + m))$ time.

$i = 4$: Let us describe a procedure that, given an arc sa' and a vertex $d' \notin \{s, a'\}$, checks whether there is an E_4-subdivision S with a-vertex a', d-vertex d', and such that $sa' \in A(S)$. Such a subdivision is said to be (sa', d')-*forced*.

We check with a Menger algorithm whether there are three independent $(a', \{s, d'\})$-dipaths, where two of the paths end up at d' and one at s. If three such dipaths do not exist, then there is clearly no (sa', d')-forced E_4-subdivision, and we return 'no'. If three such dipaths exist, then their union together with the arcs sa' form an (sa', d')-forced E_4-subdivision.

Applying the above procedure for all possible pairs (sa', d'), one solves E_4-SUBDIVISION in $O(mn(n+m))$ time.

$i = 5$: Let us describe a procedure that, given two distinct vertices b', c' and a set $S = \{s_1, s_2, s_3\}$ of three distinct inneighbours of b' checks whether there is an E_5-subdivision S' with b-vertex b', c-vertex c', and such that $\{s_1b', s_2b', s_3b'\} \subset A(S')$. Such a subdivision is said to be (b', c', S)-*forced*.

We check with a Menger algorithm, if there are three independent (c', S)-dipaths P_1, P_2, P_3, and we check whether there is a $(b', S \setminus \{c'\})$-dipath Q in $D - c'$. If four such dipaths do not exist, then we return 'no' because there is no (b', c', S)-forced E_5-subdivision. If such dipaths P_1, P_2, P_3 and Q exist, then let x be the first vertex of Q in $P_1 \cup P_2 \cup P_3$. Then the union of $P_1, P_2, P_3, Q[b', x]$ and the three arcs s_1b', s_2b', s_3b' form a (b', c', S)-forced E_5-subdivision.

Applying the above procedure for all possible triples (a', b', S), one solves E_5-SUBDIVISION in $O(n^5(n + m))$ time.

$i = 6$: Observe that every E_6-subdivision may be seen as an E_6-subdivision in which the arc cd is not subdivided. Henceforth, by an E_6-subdivision, we mean such a subdivision.

Let us describe a procedure that, given two disjoint arcs, sb' $d'c'$, returns 'yes' if it finds an E_6-subdivision and returns 'no' only if there is no E_6-subdivision S with b-vertex b', c-vertex c', d-vertex d' and such that $\{sb', d'c'\} \subseteq A(S)$. Such a subdivision is called $(sb', d'c')$-*forced*.

Applying a Menger algorithm, we check whether in D there are three independent $(b', \{s, c', d'\})$-dipaths P_1, P_2, P_3 with $t(P_1) = s$ and applying a search we check whether there is a (c', s)-dipath Q in $D - \{b', d'\}$. Clearly, if four such dipaths do not exist, then D contains no $(sb', d'c')$-forced E_6-subdivision, so we return 'no'. Conversely, if these dipaths exist, then Q contains a (c', P_1)-subdipath R. Let c'' be the last vertex along R in $V(P_2 \cup P_3)$. Now in $P_2 \cup P_3 \cup R[c', c''] \cup d'c'$, there are two internally disjoint (b', c'')-dipaths P'_2, P'_3. Thus $P_1 \cup sb' \cup P'_2 \cup P'_3 \cup R[c'', t(R)]$ is an E_6-subdivision, and we return 'yes'.

Doing this for every possible pair $(sb', d'c')$, one decides in $O(m^2(n + m))$ time whether D contains an E_6-subdivision.

$i = 7$: We proceed in two stages. We first check whether there is an E_7-subdivision in which the arc ab is not subdivided. Next we check whether there is an E_7-subdivision in which the arc ab is subdivided.

In the first stage we decide whether there is an E_7-subdivision with a-vertex a' and b-vertex b' for some arc $a'b'$. To do so, for every dipath $a'uv$ in $D - b'$, we check whether there is an E_7-subdivision with a-vertex a' and b-vertex b', and which contains the arcs of $\{a'u, uv, a'b'\}$. Such a subdivision is said to be $(a'uv, a'b')$*-forced*.

We proceed as follows. Applying a Menger algorithm, we check whether in $D - u$ there are independent $(\{v, b'\}, a')$-dipaths P_1 and P_2 with $s(P_1) = v$, and applying a search we check whether there is a (v, b')-dipath Q in $D - a' - u$. Clearly, if three such dipaths do not exist, then D contains no $(a'uv, a'b')$-forced E_7-subdivision, so we return 'no'. Conversely, if these dipaths exists, then Q contains a (P_1, P_2)-subdipath R. Then the union of P_1, P_2, R, $a'uv$, and $a'b'$ is an E_7-subdivision, and we return 'yes'. Doing this for every possible pair $(a'uv, a'b')$, one decides in $O(m^2(n + m))$ time that either D contains an E_7-subdivision, or that D contains no E_7-subdivision in which the arc ab is not subdivided.

Let G_7 be the digraph obtained from E_7 by subdividing the arc ab into a dipath awb of length 2. The second stage consists in deciding whether D contains an G_7-subdivision. We use a procedure similar to the one for detecting superstar subdivision. Given a pair $\{a'w_1x_1, a'w_2x_2\}$ of dipaths that are disjoint except for their initial vertex a', and two distinct inneighbours y_1, y_2 of a' that are not in $\{w_1, w_2\}$ (allowing the possibility that $\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$), the procedure returns 'yes' if it finds an G_7-subdivision and returns 'no' only if there is no G_7-subdivision with a-vertex a' containing all arcs in $A' = \{a'w_1, w_1x_1, a'w_2, w_2x_2, y_1a', y_2a'\}$. Such a subdivision is called A'*-forced*.

The procedure proceeds as follows. With a Menger algorithm, we first check whether in $D - \{a', w_1, w_2\}$ there are two disjoint dipaths P_1, P_2 from $\{x_1, x_2\}$ to $\{y_1, y_2\}$. If not, then D certainly does not contain any A'-forced G_7-subdivision. If yes, then check whether there is a (P_1, P_2)-dipath Q in $D - \{a', w_1, w_2\}$. If such a dipath exists, then the union of the paths $P_1, P_2, Q, a'w_1x_1, a'w_2x_2$ and the arcs y_1a' and y_2a' is an G_7-subdivision and we return 'yes'. Next, we check if there is a (P_2, P_1)-dipath Q in $D - \{a', w_1, w_2\}$. If Q exists, we return 'yes'. If not, then no A'-forced G_7-subdivision exists, because there is no vertex $x \in \{x_1, x_2\}$ with two vertices of $\{y_1, y_2\}$ in its outsection in $D - \{a', w_1, w_2\}$. So we return 'no'.

This procedure runs in linear time. Thus, running it for every possible set A', one decides in $O(m^2n^3(n+m))$ time whether D contains an G_7-subdivision, which is nothing but an E_7-subdivision in which the arc ab is subdivided.

Doing the two stages one after another, we obtain an $O(m^2n^3(n + m))$-time algorithm for solving E_7-SUBDIVISION.

$i = 8$: Similarly to the case $i = 7$, we proceed in two stages. We first check whether there is an E_8-subdivision in which the arc ab is not subdivided. Next we check whether there is an E_8-subdivision in which the arc ab is subdivided.

The first stage is the following. For every vertex a', every two distinct outneighbours b', u, and every inneighbour t' of a' distinct from b' and u, we run a procedure that returns 'yes' if it finds an E_8-subdivision, and return 'no' if there is no E_8-subdivision with a-vertex a' and b-vertex b' and whose arc set includes $\{t'a', a'b', a'u\}$. Such a subdivision is called $(t'a', a'b', a'u)$-*forced*. The procedure is the following. With a Menger algorithm, we check whether in $D-u$ there are two independent $(b', \{a', t'\})$-dipaths P_1, P_2 and whether there is a (u, t')-dipath Q in $D-\{a', b'\}$. If three such paths do not exist, then D certainly contains no $(t'a', a'b', a'u)$-forced E_8-subdivision and we return 'no'. If these three paths exist, we then we return 'yes'. Indeed let d' be the first vertex along Q in $P_1 \cup P_2$. Now the union of P_1, P_2, $Q[u, d']$, $a'b'$, $t'a'$ and $a'u$ is an E_8-subdivision with a-vertex a' and b-vertex b'.

Doing this for every possible triple $(t'a', a'b', a'u)$, one can decide in time $O(n^2m(n+m))$ whether there is an E_8-subdivision with in which the arc ab is not subdivided.

Observe that an E_8-subdivision in which ab is subdivided is an G_7-subdivision. Hence the second phase is exactly the same as the one for E_7.

Doing the two stages one after another, we obtain an $O(m^2n^3(n+m))$-time algorithm for solving E_8-SUBDIVISION. □

5.2 E_9 is tractable

Theorem 29. E_9-SUBDIVISION *can be solved in* $O(n^7(n+m))$ *time.*

The proof relies on the following notion. A *shunt* is a digraph composed of three dipaths P, Q and R such that R has length at least 2, $s(R) \in P$, $t(R) \in Q$ and P, Q, R^0 are disjoint. We frequently refer to a shunt by the triple (P, Q, R). An (S, T)-*shunt* is a shunt (P, Q, R) such that $\{s(P), s(Q)\} = S$ and $\{t(P), t(Q)\} = T$.

We consider the following decision problem.

SHUNT
Input: A digraph D and four distinct vertices s_1, s_2, t_1, t_2.
Question: Does D contain an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt?

Assume that there are two disjoint dipaths P, Q from $\{s_1, s_2\}$ to $\{t_1, t_2\}$ in D. We now give some necessary and sufficient conditions considering P and Q for D to have an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.

For any vertex x in $V(P)$, an x-*bypass* is a dipath B internally disjoint from P and Q with initial vertex in $P[s(P), x[$ and terminal vertex in $P]x, t(P)]$. Similarly, for any vertex x in $V(Q)$, an x-*bypass* is a dipath B internally disjoint from P and Q with initial vertex in $Q[s(Q), x[$ and terminal vertex in $Q]x, t(Q)]$. If x is the end-vertex of an arc between P and Q, then every x-bypass is said to be an *arc bypass*. A *crossing* (with respect to P and Q) is a pair of arcs $\{uv, u'v'\}$ such that u is before v' along P and u' is before v along Q. If uv' is an arc of P and $u'v$ is an arc of Q, then the crossing is *tight*. Otherwise it is *loose*.

Let $C = \{uv, u'v'\}$ be a tight crossing. A C-*forward path* is a dipath internally disjoint from P and Q either with initial vertex u and terminal vertex v', or with initial vertex u' and terminal vertex v. A C-*backward path* is a dipath internally disjoint from P and Q either with initial vertex in $P[v', t(P)]$ and terminal vertex in $P[s(P), u]$, or with initial vertex in $Q[v, t(Q)]$ and terminal vertex in $Q[s(Q), u']$. A C-*backward arc* is an arc that forms a C-backward path of length 1. A C-*bypass* is an x-bypass B, where x is an endvertex of a C-backward arc and if $x \in P[s(P), u]$ (resp. $Q[s(Q), u']$), $t(B)$ is also in $P[s(P), u]$ (resp. $Q[s(Q), u']$), or if $x \in P[v', t(P)]$ (resp. $Q[v, t(Q)]$), $s(B)$ is also in $P[v', t(P)]$ (resp. $Q[v, t(Q)]$).

Lemma 30. *Let D be a digraph, and let P and Q be two disjoint dipaths from $\{s_1, s_2\}$ to $\{t_1, t_2\}$. D has an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt if and only if one of the following holds :*

(a) there is a (P, Q)-dipath or a (Q, P)-dipath R of length ≥ 2;

(b) there is an arc bypass for some arc uv between P and Q;

(c) *there is a loose crossing;*

(d) *there is a tight crossing C with a C-forward path, a backward path of length at least 2 or a crossing bypass.*

Proof. Let us first price that if one of (a)–(e) holds, then D has an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.

(a) If such a dipath R exists, then (P, Q, R) or (Q, P, R) is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.

(b) If B is a u-bypass and $u \in V(P)$, then $(P[s(P), s(B)] \cup B \cup P[t(B), t(P)], Q, P[s(B), u] \cup uv)$ is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. There is a shunt constructed analogously if $u \in V(Q)$ and also when B is a v-bypass.

(c) Let $\{uv, u'v'\}$ be a loose crossing. By symmetry, we may assume that uv' is not an arc. Then $(P[s(P), u] \cup uv \cup Q[v, t(Q)], Q[s(Q), u'] \cup u'v' \cup P[v', t(P)], P[u, v'])$ is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.

(d) Let $C = \{uv, u'v'\}$ be a tight crossing.

If there is a C-forward path, then replacing the arc uv' on P or the arc $u'v$ on Q by this C-forward path, we obtain two dipaths with a loose crossing, so we are done by (c).

If there is a C-backward path R of length at least 2, then $P[s(P), u] \cup uv \cup Q[v, t(Q)], Q[s(Q), u'] \cup u'v' \cup P[v', t(P)])$ and R form an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.

Suppose now that B is a C-bypass. By symmetry and directional duality, we may assume that B is an x-bypass with $t(B) \in P[s(P), u]$. Let $a = xw$ be the corresponding C-backward arc a, where $w \in P[v', t(P)]$. Then $(Q[s(Q), u'] \cup u'v' \cup P[v', t(P)]), P[s(P), s(B)] \cup B \cup P[t(B), u] \cup uv \cup Q[v, t(Q)], wx \cup P[x, t(B)])$ is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt.

Let us now prove the reciprocal by the contrapositive. Suppose for a contradiction none of (i)-(iv) holds, but D contains an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt (P', Q', R'). Without loss of generality, we may assume that this shunt maximizes $|(A(P) \cup A(Q)) \cap (A(P') \cup A(Q'))|$. Free to swap the names of P and Q, we may assume that $s(P) = s(P')$.

Let u be the farthest vertex along P' such that $P'[s(P'), u]$ does not intersect Q. Necessarily $u \in V(P)$ for otherwise there would be a dipath of length at least 2 from P to Q. In addition, for the same reason, if $u \neq t(P)$, then the outneighbour v of u in P' must be in Q. Hence all vertices of $P'[s(P'), u] \cap P$ are in $P[s(P), u]$, for otherwise there would be a u-bypass in P, which would be an arc bypass for uv. Note also that, for every vertex x in $P[s(P), u] - P'$, there is a subdipath of P' which is an x-bypass. So $Q' \cap P[s(P), u] = \emptyset$, for otherwise in Q' there would be adipath form Q to $P[s(P), u]$ which is either has length at least 2 or is an arc with an arc bypass in P'. Let R'' be the shortest subdipath of P' with initial vertex in $V(P)$ and terminal vertex $s(R')$ if $s(R') \in P'[s(P'), u]$, and let R'' be the path of length 0 ($s(R')$) otherwise. Now, $(P'', Q'', R) = (P[s(P), u] \cup P'[u, t(P')], Q', R'' \cup R')$ is $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. Moreover if $P'[s(P'), u] \neq P[s(P), u]$, then P'' and Q'' have more arcs in common with P and Q than P' and Q', which contradicts our choice of (P', Q', R'). Therefore $P'[s(P'), u] = P[s(P), u]$.

Let u' be the farthest vertex along Q' such that $Q'[s(Q'), u']$ does not intersect P. As above, one shows that $Q'[s(Q'), u'] = Q[s(Q), u']$.

If $u = t(P)$, then $P' = P$ and necessarily $Q = Q'$. Thus R' is a dipath of length at least 2 from P to Q as (P', Q', R') is a shunt, which is a contradiction. Therefore, we may assume that $u \neq t(P)$ and similarly $u' \neq t(Q)$. Furthermore the out-neighbour v of u in P' is in $V(Q)$ and the out-neighbour v' of u' is in $V(P)$. Since P' and Q' are disjoint, $P'[s(P'), u] = P[s(P), u]$ and $Q'[s(Q'), u'] = Q[s(Q), u']$, it follows that $C = \{uv, u'v'\}$ is a crossing with respect to P and Q, and thus a tight crossing.

Consider the dipath R'.

- Assume first that $s(R') \in P'[s(P'), u]$. Let S be the shortest subdipath of $R' \cup Q'[t(R'), t(Q')]$ such that $s(S) = s(R')$ and $t(S) \in V(P) \cup V(Q)$. Vertex $t(S)$ cannot be in $Q[s(Q), u']$ for otherwise $S = R'$ and it would be a dipath of length at least 2 between P and Q. Furthermore, $\{s(R')t(S), u'v'\}$ is a loose crossing, since the distance between u' and $t(S)$ in Q is at least 2 (u is between $s(R')$ and v, and v is between u' and $t(S)$). Therefore $t(S) \in V(P)$ and so $t(S)$ is on $P[v', t(P)]$. But then S is a forward path or an arc bypass in P, a contradiction.

- Assume now that $s(R') \in P'[v, t(P')]$.

  Set $P^* = Q[s(Q), u'] \cup u'v \cup P'[v, t(P')]$ and $Q^* = P[s(P), u] \cup uv' \cup Q'[v', t(Q')]$. If $t(R') \in Q'[v', t(Q')]$, then (P^*, Q^*, R') is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. But P^* and Q^* have more arcs in common with P and Q than P' and Q', which contradicts our choice of (P', Q', R'). Therefore $t(R') \in Q'[s(Q'), u']$.

  Let S be the shortest subdipath of $P'[v, s(R')] \cup R'$ such that $t(S) = t(R')$ and $s(S) \in V(P) \cup V(Q)$.

  Assume first that $s(S) \in V(Q)$. Then S is a C-backward path. Hence it must have length 1. Therefore $s(S) \notin V(P') \cup V(Q')$ because R' has length at least 2. Let u_1 be the farthest vertex on $P'[v, t(P')]$ that is in $V(Q)$ and such that $P'[v, u_1]$ does not intersect P. Observe that u_1 appears before $s(S)$ in Q, for otherwise there would be a C-bypass in P', as $s(S) \notin P'$. In particular, u_1 is not the terminal vertex of P'. Let v_1 be the first vertex after u_1 along P' which is on $P \cup Q$. It must be in $V(P)$ by the choice of u_1. Therefore u_1v_1 is an arc because there is no dipath of length at least 2 between Q and P. Let u_2 be the farthest vertex on $Q'[v', t(Q')] \cap P$ such that $Q'[v', u_2]$ does not intersect Q. Then v_1 is after u_2 along P, for otherwise there would be an arc bypass in P for u_1v_1. Thus u_2 is not the terminal vertex of Q'. Let v_2 be the first vertex after u_2 along Q' which is on $P \cup Q$. It must be in $V(Q)$ by the choice of u_2. Hence u_2v_2 is an arc because there is no dipath of length at least 2 between P and Q. Moreover, observe that for every vertex x in $Q[v, u_1] - P'$ there is a subdipath of P' which is an x-bypass. Therefore v_2 must be in $Q]u_1, t(Q)]$ for otherwise it would be an arc bypass. Hence $\{u_2v_2, u_1v_1\}$ is a crossing for $P \cup Q$, and so it must be tight. This implies in particular that $s(S) \in Q[v_2, t(Q)]$.

  Set $P^+ = P'[s(P'), u] \cup u, v' \cup Q[v', u_2] \cup u_2v_1 \cup P'[v_1, t(P')])$ and $Q^+ = Q'[s(Q), u'] \cup u'v \cup P'[v, u_1] \cup u_1v_2 \cup Q'[v_2, t(Q')])$. If $s(R') \in P'[v_1, t(P')])$, then (P^+, Q^+, R') is an $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt. But P^+ and Q^+ have more arcs in common with P and Q than P' and Q', which contradicts our choice of (P', Q', R'). Therefore $s(R') \in P'[(v, u_1)]$. Now $P'[v, s(R')] \cup R'$ contains a subdipath T that is internally disjoint from P and Q and has initial vertex in $Q[v, u_1]$ and terminal vertex in $P \cup Q[v_2, t(Q)]$. Necessarily, $t(T) \in V(P)$ for otherwise T is an arc bypass. Hence T is an arc. Furthermore, $t(T)$ could not be in $P[v', u_2]$ for otherwise Q' would contain a $t(T)$-bypass, which would be an arc bypass. Hence $t(T) \in P]v_1, t(Q)]$ and $\{u_2v_2, T\}$ is a loose crossing, a contradiction.

  Assume now that $s(S) \in V(P)$.Then it must be in $P[v', t(P)]$. Since there is no dipath of length at least 2 from P to Q, S has length 1. Moreover, since R' has length at least 2, $s(S)$ is an internal vertex of R', so it is not in $V(P' \cup Q')$. Let u_2 be the farthest vertex on $Q'[v', t(Q')]$ that is in $V(P)$ and such that $Q'[v', u_2]$ does not intersect Q. Then u_2 appears before $s(S)$ on P, for otherwise there would be an arc bypass for $s(S)t(S)$ in P and so u_2 is not the terminal vertex of Q'. Let v_2 be the first vertex after u_2 along Q' which is on $P \cup Q$. It must be in $V(Q)$ by the choice of u_2, and so on $Q[v, t(Q)]$. u_2v_2 is an arc for otherwise for otherwise there would be a dipath of length 2 from P to Q. Let u_1 be the farthest vertex on $P'[v, t(P')]$ that is also in $V(Q)$ such that $P'[v, u_1]$ does not intersect P. Vertex u_1 appears before v_2 in Q, for otherwise there would be an arc bypass for u_2v_2 in Q, and so u_1 is not the terminal vertex of P'. Let v_1 be the first vertex after u_1 along

P' which is on $P \cup Q$. It must be in $V(Q)$ by the choice of u_1. Hence u_1v_1 is an arc because there is no dipath of length at least 2 between Q and P. Moreover, observe that for every vertex x in $P[v', u_2] - Q'$ there is a subdipath of P' which is an x-bypass. Therefore v_1 must be in $P]u_2, t(P)]$ for otherwise it would be an arc bypass. Hence $\{u_2v_2, u_1v_1\}$ is crossing for $P \cup Q$, and so it must be tight. This implies in particular that $s(S) \in P[v_1, t(P)]$.

We then find a contradiction as in the previous case by considering P^+ and Q^+.

This finishes the proof of the Lemma. □

Theorem 31. SHUNT *can be solved in* $O(n^2(n+m))$ *time.*

Proof. We describe a procedure $\texttt{shunt}(D, s_1, s_2, t_1, t_2)$, solving SHUNT and estimate its time complexity. The procedure then check, by a Menger algorithm, if there are two disjoint dipaths P, Q from $\{s_1, s_2\}$ to $\{t_1, t_2\}$, which runs in $O(n+m)$ time. Observe that the arcs s_1s_2 and s_2s_1 are useless, so we remove them from D if they exist. Then we should check if there are paths of length at least 2, arc bypasses, loose crossings, C-forward paths, backward paths of length at least 2 or crossing bypasses with respect to P and Q, according to Lemma 30. For every vertex $u \in P$ (and any vertex in Q, similarly), we do the following: if u has a neighbour in Q, we test if there is a path from $P[s(P), u[$ to $P]u, t(P)]$, which would be an arc bypass. Let v' be the last vertex of Q such that uv' is an arc (and such that $v'u$ is an arc, similarly). Then, for a vertex v in $P]u, t(P)]$, we check if there is a vertex u' in $Q[s(Q), v'[$ such that $u'v$ (vu') is an arc. Then if u, v and $u'v'$ have distance at least 2 in P and Q respectively, it would be a loose crossing. Otherwise, if such edges exists there is a tight crossing $C = \{uv', u'v\}$ containing u. We then run a Menger algorithm one more time, to test if there is a dipath from u to v in $D - P - Q$, which would be a forward path. So far, the running time of the algorithm is bounded by $O(n^2(n+m))$: the complexity of calculating the P and Q initially plus the complexity of, for each vertex in in $P \cup Q$, look for an arc bypass, plus the running time of analysing if each pair of vertices in P or Q are part of a loose crossing and finally plus the time of looking for a forward path. Then, still considering the same tight crossing C, for every vertex x in $P[v, t(P)]$, we check if there is a dipath to some y in $P[v, t(P)]$. If it is the case and xy is an arc, we then look for dipaths from $P[s(P), y[$ to $P]y, u]$ and from $P[v, x[$ to $P]x, t(P)]$. This can be done in $O(n^2(n+m))$: for every pair of vertices u and x, we uses Menger algorithm possibly three times to compute the dipaths above. So, $\texttt{shunt}(D, s_1, s_2, t_1, t_2)$ runs in $O(n^2(n+m))$ time in total. □

With Theorem 31 in hands, we now deduce Theorem 29. We believe that it could also be used to prove some other digraphs F to be tractable.

Proof of Theorem 29. For every vertex v of D and for every set of two outneighbours s_1, s_2 and two inneighbours t_1, t_2 of v, we check if there is a $(\{s_1, s_2\}, \{t_1, t_2\})$-shunt in D. Observe that there is an E_9-SUBDIVISIONin D in which v is the a-vertex if and only if there is a shunt for a pair of outneighbours and a pair of inneighbours of v. So, since there are n^5 possible choices for vertex v and its neighbours, and for each of them we apply the procedure $\texttt{shunt}$ that runs in $O(n^2(n+m))$ time, our algorithm decides whether there is an E_9-SUBDIVISION in D in $O(n^7(n+m))$ time.

□

6 Towards a full classification of digraphs of order 4

In this section, we review all digraphs D of order 4, and determine if they are tractable or hard or if their status is unknown.

For a digraph D, its 2-*cycle graph* G_D is the graph with the same vertex set in which two vertices are linked by an edge if they are in a directed 2-cycle in D. Thus, the 2-cycle graph of an oriented graph is an empty graph. We denote by $A'(D)$ be the set of arcs of D which are not in directed 2-cycles.

Let F be a digraph of order 4. By Corollary 10, if F contains a directed 2-cycle whose vertices are big, then F is hard. So we may assume that F contains no such 2-cycles. In particular, it implies that G_F has at most one vertex of degree at least two. So G_F has at most three edges.

Case 0: G_F has no edges. Then F is tractable by Theorem 17.

Case 1: G_F has three edges. Then necessarily, G_F is the star of order 3. Hence F is either the symmetric star or the superstar of order 4. In both cases, F is tractable, see Subsection 5.1.

Case 2: G_F has exactly two edges which are non-adjacent.

If $|A'(F)| \leq 1$, then F has no big vertex, so by Corollary 3 F is tractable.

If $|A'(F)| \geq 2$, then F is either one of N_1, N_2, N_3, N_4, O_1 and their converses, or F has no big vertex. In the later case, F is tractable by Corollary 3. If $F = N_i$ for $i \in \{1, 2, 3, 4\}$, then F is hard by Proposition 25. We do not know the complexity of O_1-subdivision.

Case 3: G_F has exactly two edges which are adjacent.

If $A'(F)$ is empty, then $F = SS_2 + K_1$, where K_1 is the digraph on one vertex. As discussed in Subection 5, SS_2 is tractable. Thus, by Lemma 12, F is tractable.

If $|A'(F)| = 1$, then F either is $SS_2^* + K_1$, or E_1 or the converse of E_1, or is obtained from SS_2^* by gluing an arc on its centre. Now $SS_2^* + K_1$ is tractable by Corollary 27 and Lemma 12; E_1 (and thus its converse) is tractable by Proposition 28; if F is obtained from SS_2^* by gluing an arc on its centre, then it is tractable by Theorem 26 and by Lemma 13.

If $|A'(F)| = 2$, then F is either E_2, E_3, E_9, O_2 or one of their converses. If $F \in \{E_2, E_3, E_9\}$, then it is tractable by Proposition 28. If $F = O_2$, then we do not know.

If $|A'(F)| = 3$, then F is either N_5, N_6, O_3 or one of their converses. If $F \in \{N_5, N_6\}$, then it is hard by Proposition 25. The complexity of O_3-SUBDIVISION is still unknown.

Case 4: G_F has exactly one edge.

If F has no big vertices, then, by Corollary 3, F is tractable. Henceforth, we may assume that F has a big vertex, i.e. a vertex with in-degree or out-degree at least 3. Observe that it implies that F is connected and $|A'(F)| \geq 2$.

$|A'(F)| = 2$, then either F obtained from $\vec{C_2}$ by gluing a spider on one its vertices, or it is one of the E_i for $i = 4, 5$ or one of their converses. In each of these cases, F is tractable by Lemma 13 or by Proposition 28.

If $|A'(F)| = 3$, then we distinguish several subcases according to the position of the arcs of $A'(F)$ relatively to the directed 2-cycle C of F.

- $A'(F)$ induces an orientation of a star. Then F is obtained from W_2 or its converse by gluing an arc on its centre. Thus F is tractable by Lemma 13.
- $A'(F)$ induces an oriented path whose first vertex is a vertex of C and whose third vertex is the other vertex of C. Then F is obtained either from the bispindle $B(2, 1; 1)$ by gluing an arc on one of its nodes, or from W_2 or its converse by gluing an arc on one of its vertices. In both cases, F is tractable by Lemma 13 and Theorem 15 and Lemma 11.
- $A'(F)$ induces an oriented 3-cycle. If this cycle is directed, then F is a *windmill*, that is a subdivision of a symmetric stars. Bang-Jensen et al. [2] proved that windmills are tractable, so F is tractable. If this cycle is not directed, then F is either E_4 or its converse, or N_7. If F is E_4 or its converse, then it is tractable by Proposition 28. If $F = N_7$, then it is hard by Proposition 25.

If $|A'(F)| = 4$, then it is either N_8, N_9, E_5, E_6, E_7, E_8, O_4, O_5, or one of their converses. If F is N_8 or N_9, then it is hard by Proposition 25. If F is E_5, E_6, E_7, or E_8, then it is tractable by Proposition 28. The complexity of O_4-SUBDIVISION and O_5-SUBDIVISION is still open.

References

[1] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer Monographs in Mathematics. Springer, 2008.

[2] J. Bang-Jensen, F. Havet, and A. K. Maia de Oliveira Finding a subdivision of a digraph *Theoretical Computer Science*, 562:283–303, 2015.

[3] D. Berwanger, A. Dawar, P. Hunter, and S. Kreutzer. DAG-width and parity games. In STACS 2006, Proc. 23rd Symposium on Theoretical Aspects of Computer Science, Volume 3884 of Lect. Notes Comput. Sci., pages 524-536. Springer-Verlag, 2006.

[4] J.A. Bondy and U.S.R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, 2008.

[5] M. Chudnovsky, A. Scott and P.D. Seymour. Disjoint paths in tournaments. Manuscript 2012.

[6] L.R. Ford Jr. and D. R. Fulkerson. *Flows in Networks*. Princeton University Press, Princeton, NJ, 1962.

[7] S. Fortune, J.E. Hopcroft, and J. Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10:111–121, 1980.

[8] F. Havet, and A. K. Maia. On disjoint directed cycles with prescribed minimum lengths. *Technical report* RR-8286, INRIA, April 2013.

[9] T. Johnson, N. Robertson, P.D. Seymour and R. Thomas. Directed tree-width. *J. Combinatorial Theory, Ser. B* , 82:138–154, 2001.

[10] K. Kawarabayashi et S. Kreutzer. The directed grid theorem. arXiv:1411.5681 [cs.DM]

[11] Intercyclic digraphs. *Graph Structure Theory*, (Neil Robertson and Paul Seymour, eds.), *AMS Contemporary Math.*, 147:203–245, 1993.

[12] B. Reed, N. Robertson, P.D. Seymour, and R. Thomas. Packing directed circuits. *Combinatorica*, 16(4):535–554, 1996.

A Solving F-subdivision without using the Directed Grid Theorem

The tractability of some digraphs is given by Corollary **??** and so derives from the Directed Grid Theorem. In this appendix, we give elementary algorithms (i.e. not requiring the Directed Grid Theorem) digraphs or order 4 with no big vertices are tractable. Such algorithms were already given for some digraphs in [2]. We present here elementary algorithms for the remaining ones. We first start by oriented graphs in Subsection **??**. Then, after giving other tools, we give the algorithms for the other digraphs of order 4 with no big vertices.

A.1 Oriented graphs

There are four oriented of order 4 with no big vertices that were not proved to be tractable in [2]. They are depicted in Figure 7.

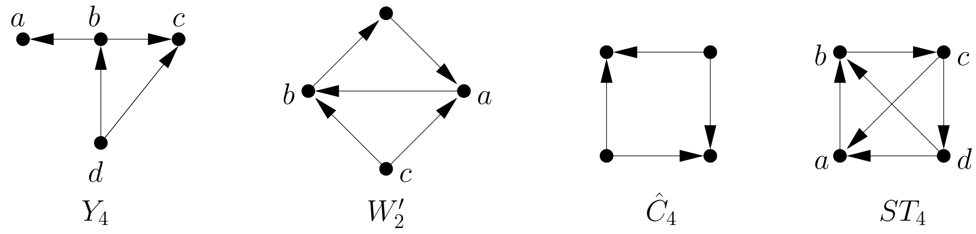


Figure 7: The oriented graphs Y_4, W_2', $\hat{C}_4$, and ST_4

A.1.1 Y_4-subdivision

Theorem 32. Y_4-SUBDIVISION *can be solved in $O(n^5(n+m))$ time.*

Proof. Let us describe a procedure that given three distinct vertices a', c', d', and two distinct arcs $d'u_1$ and $d'u_2$ in $D - a'$ decides whether a digraph D contains a Y_4-subdivision S with a-vertex a', c-vertex c', d-vertex d' such that $\{d'u_1, d'u_2\} \subseteq A(S)$. Such a subdivision is said to be $(a', c', d'u_1, d'u_2)$*-forced.*

We check whether there are two independent $(\{u_1, u_2\}, c')$-dipaths P_1, P_2 in $D - \{a', d'\}$ and a $(\{u_1, u_2\}, a')$-dipath Q in $D - \{c', d'\}$. This can be done in linear time using a Menger algorithm for each of the tasks. The existence of P_1, P_2, Q is clearly a necessary condition to contain an $(a', c', d'u_1, d'u_2)$-forced Y_4-subdivision. So if we do not find such dipaths, we return 'no'. If we have such dipaths, then we return 'yes'. Indeed the union of the dipaths $d'u_1, d'u_2, P_1, P_2$, and R, where R is the $(P_1 \cup P_2, a')$-subdipath of Q, is an $(a', c', d'u_1, d'u_2)$-forced Y_4-subdivision.

Doing this for every 5-tuple (a', c', d', u_1, u_2) of vertices, we obtain an algorithm solving Y_4-SUBDIVISION in $O(n^5(n+m))$ time. □

A.1.2 Subdivision of the antidirected cycle of length 4

Theorem 33. $\hat{C}_4$-SUBDIVISION *can be solved in $O(n^3(n+m))$ time.*

Proof. Observe that a $\hat{C}_4$-subdivision is an oriented cycle with four blocks.

We shall describe a polynomial-time procedure $\hat{C}_4$-`Subdivision`(a, b, D) that, given two vertices a, b, either finds a $\hat{C}_4$-subdivision (not necessarily with sources a and b) and in this case returns 'yes', or verifies that there is no $\hat{C}_4$-subdivision in D with a and b as sources and returns 'no'. Since a $\hat{C}_4$-subdivision has two sources, running this procedure for every pair $\{a, b\}$ of vertices yields an algorithm to decide whether D contains a $\hat{C}_4$-subdivision; in addition the algorithm runs in $O(n^3(n+m))$ time, because the procedure $\hat{C}_4$-$\texttt{Subdivision}(a, b, D)$ only needs $O(n(n+m))$ time.

First, we determine the outsections $S_a = S^+_{D-b}(a)$ and $S_b = S^+_{D-a}(b)$. If there is a $\hat{C}_4$-subdivision with sources a and b in D, then its two sinks must be in $X = S_a \cap S_b$. Thus if $|X| \leq 1$, we return 'no'. Henceforth, we assume that $|X| \geq 2$.

Let A (resp. B) be the set of vertices $x \in X$ such that there is an (a, x)-dipath in $D - b$ (resp. (b, x)-dipath in $D - a$) whose internal vertices are not in X. If there is a $\hat{C}_4$-subdivision with sources a and b in D, then A and B must both be of size at least 2. Thus if $|A| \leq 1$ or $|B| \leq 1$, we return 'no'. Henceforth, we assume that $|A| \geq 2$ and $|B| \geq 2$.

Claim 33.1. *Let D' be the digraph obtained from $D[X]$ by adding a, b and all arcs from a to A and from b to B. Then*

(i) *if D has a $\hat{C}_4$-subdivision with sources a and b, then so does D';*

(ii) *if D' has a $\hat{C}_4$-subdivision, then so does D.*

Subproof. (i) Assume that D contains a $\hat{C}_4$-subdivision S with sources a and b, and let c and d be the sinks of S. Let P_1 (resp. P_2, Q_1, and Q_2) be the (a, c)-dipath, (resp. (a, d)-dipath, (b, c)-dipath, and (b, d)-dipath) in S and let a_1 (resp. a_2, b_1, b_2) be the last vertex of A (resp. A, B, and B) on this path. Observe that $V(P_1[a_1, c]) \subseteq X$ and that a similar property holds for each of the paths $P_2[a_2, d]$, $Q_1[b_1, c]$, and $Q_2[b_2, d]$. This shows that the digraph which is the union of the four dipaths $aa_1P_1[a_1, c]$, $aa_2P_2[a_2, d]$, $bb_1Q_1[b_1, c]$ and $bb_2Q_2[b_2, d]$ is a $\hat{C}_4$-subdivision with sources a and b in D'.

(ii) Suppose that D' has a $\hat{C}_4$-subdivision S'. If $a \in V(S')$, let a_1 and a_2 be the two outneighbours of a in S'. Clearly, $a_1, a_2 \in A$. Therefore in $D - b$, there exist an (a, a_1)-dipath P_1 and an (a, a_2)-dipath whose internal vertices are not in X. Let a' be the last vertex in $P_1 \cap P_2$ on P_1. We set $P' = \overleftarrow{P_2}[a_1, a']P_2[a', a_2]$.

Similarly, if $b \in V(S')$, denoting b_1 and b_2 the two outneighbours of b in S', one can find a (b_1, b_2)-inpath with two blocks whose internal vertices are not in X. Call this path Q'.

Now replacing in S' the oriented path a_1aa_2 by P' if $a \in V(S')$ and the oriented path b_1bb_2 by Q' if $b \in V(S')$ results in a $\hat{C}_4$-subdivision in D. $\Diamond$

By Claim 33.1, we can replace D by D', i.e. we may assume henceforth that $D = D'$, $X = V(D) \setminus \{a, b\}$, $A = N^+(a)$ and $B = N^+(b)$. Moreover, we will assume that $N^-(a) = N^-(b) = \emptyset$.

If $|A \cap B| \geq 2$, then we return 'yes'. Indeed, for any two distinct vertices c and d in $A \cap B$, the cycle $acbda$ is isomorphic to $\hat{C}_4$. Therefore, we may assume that $|A \cap B| \leq 1$.

If $|A \cap B| = 1$, say $A \cap B = \{d\}$, then we check with a Menger algorithm for each vertex $c \in V(D) \setminus \{a, b, d\}$, whether there are independent $(\{a, b\}, c)$-dipaths. If there is a vertex c with two such dipaths P and Q, then we return 'yes'. Otherwise, then we return 'no'. This is valid by the following claim.

Claim 33.2. *If $A \cap B = \{d\}$, then D contains a $\hat{C}_4$-subdivision with sources a and b if and only if there is a vertex $c \in V(D) \setminus \{a, b, d\}$ such that $D - d$ contains two independent $(\{a, b\}, c)$-dipaths.*

Subproof. If D contains a $\hat{C}_4$-subdivision S with sources a and b, then one of two oriented (a, b)-paths, say R, forming S does not contain d. Thus the sink in R is the desired vertex c.

If there is a vertex c as described above, then let P and Q be two independent $(\{a, b\}, c)$-dipaths with respective initial vertex a and b. Then $P\overleftarrow{Q}bda$ is a $\hat{C}_4$-subdivision. $\Diamond$

Assume now that $A \cap B = \emptyset$. We take a shortest (a, B)-dipath P_a (this can be in done in linear time by Breadth-First Search). Such a path exists because X is the outsection of a in $D - b$. Let c be the terminal vertex of P_a. We then search for a shortest $(a, B \setminus \{c\})$-dipath in $D - c$. If we find such a path Q_a with terminal vertex d, then we return 'yes'. Indeed denoting by a' the last vertex in $P_a \cap Q_a$ on Q_a, the oriented cycle $P_a[a', c]cbd\overleftarrow{Q}_a[d, a']$ is a $\hat{C}_4$-subdivision.

Hence we may assume that every (a, B)-dipath goes through c. Let b' be a vertex in $B \setminus \{c\}$, and let D^* be the digraph obtained by contracting $\{b, b'\}$ into a vertex b^* and removing all arcs entering b^*. We return $\hat{C}_4$-`Subdivision`(a, b^*, D^*). This is valid by Claim 33.3.

Claim 33.3. (i) *If there is a $\hat{C}_4$-subdivision with sources a and b in D, then there is a $\hat{C}_4$-subdivision with sources a and b^* in D^*.*

(ii) *If there is a $\hat{C}_4$-subdivision in D^*, then there is a $\hat{C}_4$-subdivision in D.*

Subproof. (i) Assume there is a $\hat{C}_4$-subdivision with sources a and b in D. Let S be such a subdivision with minimum number of vertices. Let b_1 and b_2 be the two outneighbours of b in S.

If $b' \notin V(S)$, then the digraph obtained from S by replacing the vertex b and the arcs bb_1 and bb_2 by the vertex b^* and the arcs b^*b_1, b^*b_2 is a $\hat{C}_4$-subdivision in D^*.

Suppose now that $b' \in V(S)$. Then bb' is an arc of S. Indeed if it were not, then replacing the (b, b')-path in S not containing a by the arc bb', we would obtain a smaller $\hat{C}_4$-subdivision with sources a and b. Thus, we may assume that $b' = b_1$.

Now b' is not a sink in S. Indeed suppose it were. Let Q be the (a, b')-dipath in S. Necessarily, Q goes through c. Thus, the digraph obtained from S by replacing Q by $Q[a, c]$ and bb' with bc is a smaller $\hat{C}_4$-subdivision with sources a and b, a contradiction.

Hence, b' has an outneighbour b'' is S. Then the digraph obtained from S by replacing the vertices b and b' and the arcs bb', $b'b''$ and bb_2 by the vertex b^* and the arcs b^*b'', b^*b_2 is a $\hat{C}_4$-subdivision in D^* with sources a and b^*.

(ii) Assume that S^* is a $\hat{C}_4$-subdivision in D^*. If b^* is not a vertex of S^*, then S^* is contained in D and we have the result. If b^* is a vertex in S^*, then it is a source since its indegree in D^* is zero. Let s and t be its two outneighbours in S^*. By definition of D^*, s and t are both in $N^+_D(b) \cup N^+_D(b')$. If s and t are both in $N^+_D(b)$ (resp. $N^+_D(b')$), then the digraph obtained from S^* by replacing the vertex b^* and the arcs b^*s and b^*t by the vertex b (resp. b') and the arcs bs and bt (resp. $b's$ and $b't$) respectively, is a $\hat{C}_4$-subdivision in D. If $s \in N^+_D(b)$ and $t \in N^+_D(b')$, then the digraph obtained from S^* by replacing the vertex b^* and the arcs b^*s and b^*t by the vertices b, b' and the arcs bs, bb' and $b't$ is a $\hat{C}_4$-subdivision in D. $\Diamond$

Let us now estimate the time complexity of $\hat{C}_4$-`Subdivision`. It first computes two outsections, which can be done in linear time. Then either it leads a recursive call or it does not because it stops. In the preparation of a recursive call, it possibly computes a dipath (in the case $A \cap B = \emptyset$). Moreover, the order of digraph decreases by one in the call. In the second case, either it stops for some easy reason in $O(1)$ steps, or it stops after using a Menger algorithm which runs in linear time. Let r be the number of recursive calls made by the $\hat{C}_4$-`Subdivision`. Clearly $r \leq n$ and the procedure runs in $O(r \times (n + m) + (n + m))$ time, that is in $O(n(n + m))$ time. $\square$

A.1.3 Subdivision of W'_2

In this subsubsection, we consider W'_2-SUBDIVISION. Using Lemmas 11 and 16, one can easily give a polynomial-time algorithm to solve it.

Theorem 34. *W'_2-SUBDIVISION can be solved in $O(n^5(n + m))$ time.*

Proof. By Lemma 11, applied to every vertex c' in D, for every vertex a' of D, one can decide in $O(n^3(n + m))$ if there is a W_2-subdivision with b-vertex b'. Observe that W'_2 is obtained from W_2 by subdividing once the arc ba, which is the only arc leaving b. Therefore, Lemma 16 applies. Thus for every vertex b' one can decide in $O\left(\sum_{y\in N^+(b')} d^+(y)n^3(n+m)\right)$ if there is a W_2'-subdivision with b-vertex b'. Hence, one can decide in $O(n^5(n+m))$ time whether there is a W_2'-subdivision in D. □

We now give a more complicated but faster algorithm based on an algorithm deciding if there is a W_2'-subdivision with prescribed c-vertex. This proof uses in a simpler way the technique that we use in Subsection 3.1 for proving that W_3 is tractable.

Theorem 35. *Given a digraph D and a vertex v of D, one can decide in $O(n^3(n+m))$ time if D contains a W_2'-subdivision with centre v. So W_2'-SUBDIVISION can be solved in $O(n^4(n+m))$ time.*

The proof of this theorem relies on the following notion. Let X be a set of two vertices. An *X-bipod* is a digraph which is the union of a directed cycle C of length at least 3 and two disjoint dipaths P_1 and P_2 with initial vertices in X and terminal vertices in C. If the P_i are (X,C)-dipaths, we say that the bipod is *unfolded*. Note that the dipaths P_i may be of length 0. We often denote a bipod by the triple (C,P_1,P_2) described above.

Proposition 36. *Let $X=\{x_1,x_2\}$ be a set of two distinct vertices. Any X-bipod contains an unfolded X-bipod.*

Proof. Let (C,P_1,P_2) be an X-bipod, where P_i has initial vertex x_i, for $i=1,2$. Let y_i be the first vertex on C along P_i. Then $(C,P_1[x_1,y_1],P_2[x_2,y_2])$ is an unfolded X-bipod. □

We shall consider the following decision problem.

BIPOD
Input: A strong digraph D and a set X of two distinct vertices of D.
Question: Does D contain an X-bipod?

Lemma 37. BIPOD *can be solved in $O(n(n+m))$ time.*

Proof. Let us describe a procedure `bipod`(D,X), solving BIPOD.

We first look for a directed cycle of length at least 3 in D. This can be done in linear time. If there is no such cycle, then we return 'no'.

Otherwise we have a directed cycle C of length at least 3. We choose a set Y of two vertices in C and run a Menger algorithm between X and Y. If this algorithm finds two disjoint (X,Y)-dipaths P_1,P_2, then we return the bipod (C,P_1,P_2). Otherwise, the Menger algorithm finds a 1-separation (W,S,Z) of (X,Y). Note that $|S|=1$ because D is strong. Set $S=\{s\}$.

Let D' be the digraph obtained from D by contracting Z into a vertex t. Note that D' is strong. We now make a recursive call to `bipod`(D',X). This is valid by virtue of the following claim.

Claim 37.1. *There is an X-bipod in D if and only if there is an X-bipod in D'.*

Subproof. Suppose first that there is an X-bipod in D'. Then D' contains an unfolded X-bipod B' by Proposition 36. If B' is contained in D, then we are done. So we may assume that it is not. Then B' contains a dipath stw for some $w\in W$. It contains only one such dipath since every vertex has outdegree at most one in B'. Moreover, t has indegree 1 in D', so it has indegree 1 also in B'. Since t was obtained by contraction of Z, w has an inneighbour $z\in Z$. Now, since D is strong, there is an (s,z)-dipath Q in D. Because there is no arc from W to Z, all the internal vertices of Q are in Z. Hence the digraph B obtained from B' by replacing the dipath stw by the dipath Qzw is an X-bipod in D.

Suppose now that D contains an X-bipod. Then it contains an unfolded X-bipod $B=(C,P_1,P_2)$ by Proposition 36. Since all (X,Z)-dipaths in D go through s, the terminal vertices of the P_i are in $W\cup S$, and $D[Z]\cap B$ is a dipath Q which is a subpath of one of the P_i or C. If Q is a (u,z)-dipath, then B

contains arcs su and zw for some $w \in W$. Then the digraph B' obtained from B by replacing $suQzw$ by the dipath stw is an X-bipod in D'. Indeed, if Q was a subpath of C, then the directed cycle in B' has length at least 3, as it contains the three vertices s, t and w. $\Diamond$

Each time we do a recursive call, the number of vertices decreases. So we do at most n of them. Between two recursive calls, we search for a directed cycle of length at least 3 and run a Menger algorithm. Both can be done in linear time. So `bipod` runs in $O(n(n+m))$ time. $\square$

With Lemma 37 in hands, we now deduce Theorem 35.

Proof of Theorem 35. Let v be a vertex of D. Let us describe an algorithm that decides whether there is a W'_2-subdivision with centre v in D. Observe that such a subdivision S is the union of a directed cycle C of length at least 3, and two internally disjoint (v, C)-dipaths P_1, P_2 with distinct terminal vertices y_1, y_2. Since it is strong, the cycle C is contained in some strong component Γ of $D - v$. For $i = 1, 2$ let x_i be the first vertex of P_i in Γ. Set $X = \{x_1, x_2\}$. Then the paths $P_i[x_i, y_i]$, $i = 1, 2$, and C form an X-bipod in Γ, and the $P_i[v, x_i]$, $i = 1, 2$, are independent (v, X)-dipaths in $D - (\Gamma \setminus X)$. Hence for finding a W'_2-subdivision with centre v, the following procedure checks whether there is a set X as above. First, we compute the strong components of $D - v$. Next, for every subset X of two vertices in the same strong component Γ, we run a Menger algorithm to check whether there are two independent (v, X)-dipaths in $D - (\Gamma \setminus X)$. If yes, we check using `bipod` whether there is an X-bipod in Γ. If yes again, then we clearly have a W'_2-subdivision with centre v, and we return 'yes'. Otherwise, there is no such subdivision, and we proceed to the next pair.

There are at most n^2 possible pairs X. And for each pair we run a Menger algorithm in $O(n+m)$ time and possibly `bipod` in $O(n(n+m))$ time. Hence our algorithm decides whether there is W'_2-subdivision with centre v in D in $O(n^3(n+m))$ time.

To solve W'_2-subdivision, we check for every vertex v in turn if there is a W'_2-subdivision with centre v. As we examine at most n vertices, this algorithm runs in $O(n^4(n+m))$ time. $\square$

A.1.4 Subdivision of ST_4

The aim of this subsubsection is to prove that ST_4 is tractable. We shall need the following two lemmas.

Lemma 38. *Let D be a digraph, C a directed cycle in D, and x a vertex in $V(D) \setminus V(C)$. If there are two (x, C)-dipaths P_1 and P_2 and a (C, x)-dipath Q such that $s(Q)$, $t(P_1)$ and $t(P_2)$ are distinct, then D contains an ST_4-subdivision.*

Proof. Assume first that P_1 and P_2 are independent. If $Q \cap (P_1^\circ \cup P_2^\circ) = \emptyset$, then $C \cup P_1 \cup P_2 \cup Q$ is an ST_4-subdivision. If Q intersects $P_1^\circ \cup P_2^\circ$, then without loss of generality, we may assume that the first vertex y along Q in $P_1^\circ \cup P_2^\circ$ is on P_1°. Let z_2 be the first vertex in $V(Q) \cap V(P_2)$ along Q. Such a vertex exists because $x \in V(Q) \cap V(P_2)$. Let z_1 be the last vertex on $Q[y, z_2]$ which is on $P_1[y, t(P_1)]$. Now $Q[z_1, z_2] \cup P_1[y, t(P_1)] \cup Q[s(Q), y] \cup P_2[z_2, t(P_2)] \cup C$ is an ST_4-subdivision.

Assume now that P_1 and P_2 are not independent. Let x' be the last vertex in $P_1 \cap P_2$ along P_1. Set $P'_1 = P_1[x', t(P_1)]$, $P'_2 = P_2[x', t(P_2)]$, and let Q' be the $(s(Q), x')$-dipath contained in the walk $QP_2[x, x']$. Then P'_1 and P'_2 are independent (x', C)-dipaths. Hence by the previous case, D contains an ST_4-subdivision. $\square$

Lemma 39. *Let D be a strong digraph, C a directed cycle in D, x a vertex in $V(D - C)$. If there are three (x, C)-dipaths with distinct terminal vertices, then D contains an ST_4-subdivision.*

Subproof. Suppose that there are three (x, C)-dipaths P_1, P_2, P_3 such that $t(P_1)$, $t(P_2)$, and $t(P_3)$ are distinct. Since D is strong, there is a (C, x)-dipath Q. Without loss of generality, we may assume that $s(Q) \notin \{t(P_1), t(P_2)\}$. Thus by Lemma 38, D contains an ST_4-subdivision. $\Diamond$

Theorem 40. ST_4-SUBDIVISION *can be solved in* $O(n^5(n+m))$ *time.*

Proof. Since ST_4 is strong, its subdivisions are also strong. So we only need to prove the result for a strong input digraph D; if the digraph is non-strong, it suffices to check whether one of its strong components contains an ST_4-subdivision.

We shall describe a procedure ST_4-$\texttt{Subdivision}(D, d')$, that, given a strong digraph D and a vertex d', returns 'no' only if there is no ST_4-subdivision in D with d-vertex d' and returns 'yes' when it finds an ST_4-subdivision (not necessarily with d-vertex d'). Running this procedure for every vertex d' yields an algorithm to decide whether D contains an ST_4-subdivision; in addition, the algorithm runs in $O(n^5(n+m))$ time, because the procedure ST_4-$\texttt{Subdivision}(D, d')$ only needs $O(n^4(n+m))$ time.

First, we check whether d' is the centre of a W_2'-subdivision. This can be done in $O(n^3(n+m))$ time, according to Theorem 35. If not, then we return 'no' since every ST_4-subdivision with d-vertex d' contains a W_2'-subdivision with centre d'.

If there is a W_2'-subdivision with centre d', let us denote by C its directed cycle, and by P_1 and P_2 the two (d', C)-dipaths in it. For $i = 1, 2$, let x_i be the terminal vertex of P_i.

Let S^- and S^+ be the insection and outsection, respectively, of d' in $D - \{x_1, x_2\}$. We compute S^- and S^+. If S^- contains a vertex in $V(C) \setminus \{x_1, x_2\}$, then there is a (C, d')-dipath Q with initial vertex $x_3 \notin \{x_1, x_2\}$. So by Lemma 38, there is an ST_4-subdivision in D, and we return 'yes'. Similarly, because of Lemma 39, we return 'yes' if S^+ contains a vertex in $V(C) \setminus \{x_1, x_2\}$.

Assume now that $(S^- \cup S^+) \cap (V(C) \setminus \{x_1, x_2\}) = \emptyset$. By the definition of outsection, no arc is leaving S^+ in $D - \{x_1, x_2\}$, so in D every arc leaving S^+ has its head in $\{x_1, x_2\}$. Similarly, all arcs entering S^- have tail in $\{x_1, x_2\}$. Moreover, because D is strong, for every vertex $s \in S^+$, there is an $(s, \{x_1, x_2\})$-dipath in $D[S^+ \cup \{x_1, x_2\}]$.

Since D is strong, there is a directed (C, d')-dipath in D. Its first arc goes from $\{x_1, x_2\}$ to S^-. Hence at least one vertex of $\{x_1, x_2\}$ has an outneighbour in S^-.

Claim 40.1. *Suppose both x_1 and x_2 have an outneighbour in S^-. If there is a (C, S^+)-dipath R with $s(R) \notin \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

Subproof. There is a $(t(R), \{x_1, x_2\})$-dipath P with internal vertices in S^+. Without loss of generality, we may assume that $t(P) = x_1$. Since x_2 has an outneighbour in S^-, there is an $(x_2, t(R))$-dipath Q whose internal vertices are in $D - C$. Hence by the directional dual of Lemma 38 (ST_4 is isomorphic to its converse), D contains an ST_4-subdivision. $\Diamond$

Each x_i has an inneighbour in P_i, and so an inneighbour in S^+. Hence a similar reasoning as the proof of Claim 40.1 gives the following.

Claim 40.2. *If there is an (S^-, C)-dipath with terminal vertex $x_3 \notin \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

For $i = 1, 2$, let S_i^+ be the set of vertices s of S^+ for which there is an (s, x_i)-dipath with internal vertices in $V(D - C)$. In the very same way as Claim 40.1, one can prove the following claim.

Claim 40.3. *Suppose x_i has no outneighbour in S^-. If there is a (C, S_i^+)-dipath with initial vertex $x_3 \notin \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

<u>Case 1:</u> Assume first that both x_1 and x_2 have an outneighbour in S^-.

Let T^+ be the outsection of S^- in $D - \{x_1, x_2\}$, T^- the insection of S^+ in $D - \{x_1, x_2\}$. The definition of T implies the following property:

(T$_1$) If $u \in V(D) \setminus (T \cup \{x_1, x_2\})$ and Q is a (u, d')-path in D with at most two blocks, then Q contains a vertex in $\{x_1, x_2\}$.

Now, we compute $T = T^- \cup T^+$. If T contains a vertex of $V(C) \setminus \{x_1, x_2\}$, then we return 'yes', since D contains an ST_4-subdivision by Claim 40.1 or 40.2. If not, then $T \cap (V(C) \setminus \{x_1, x_2\}) = \emptyset$. Let D' be the digraph obtained from $D[T \cup \{x_1, x_2\}]$ by adding the arcs x_1x_2 and x_2x_1 if they were not in $A(D)$. Observe that D' has fewer vertices than D, because the vertices of $V(C) \setminus \{x_1, x_2\}$ are not in $V(D')$ and this set is not empty because C has length at least 3. We then return ST_4-`Subdivision`(D', d'). The validity of this recursive call is established by the following claim.

Claim 40.4. *D contains an ST_4-subdivision with d-vertex d' if and only if D' does.*

Subproof. From every ST_4-subdivision in D' with d-vertex d', one can obtain an ST_4-subdivision in D with d-vertex d' by replacing the arc x_1x_2 (resp. x_2x_1) by $C[x_1, x_2]$ (resp. $C[x_2, x_1]$).

Assume now that D contains an ST_4-subdivision S with d-vertex d'. Let a', b', and c' be the vertices in S corresponding to a, b, and c, respectively.

Our first goal is to prove that $a', b', c' \in V(D')$. Let $u \in \{a', b', c'\}$ be one of these three vertices. Note that there are three internally disjoint paths in S joining u with d', and each of these paths has at most two blocks. If $u \notin V(D')$, then Property (T_1) stated above implies that each of these paths contains x_1 or x_2 as one of its internal vertices. Since the three paths are internally disjoint, this is not possible, and we conclude that $u \in V(D')$.

Hence, a', b', c', d' all belong to $V(D')$. Therefore, the intersection of S with $V(D) \setminus T$ is a dipath P whose initial vertex is dominated by $x \in \{x_1, x_2\}$ and whose terminal vertex dominates the vertex x' of $\{x_1, x_2\} \setminus \{x\}$. Hence, D' contains the ST_4-subdivision obtained from S by replacing xPx' by xx'. $\Diamond$

<u>Case 2:</u> Assume that one vertex in $\{x_1, x_2\}$, say x_1, has no outneighbour in S^-.

Let T^+ be the outsection of S^- in $D - \{x_1, x_2\}$, T_1^- the insection of S_1^+ in $D - \{x_1, x_2\}$ and $T = T^+ \cup T_1^-$. Observe that $S^+ \subseteq T$ because $d' \in S^-$. The definition of T implies the following property:

(T_2) If $u \in V(D) \setminus (T \cup \{x_1, x_2\})$ and Q is a (u, d')-path with at most two blocks, then either Q contains a vertex in $\{x_1, x_2\}$, or Q has two blocks and there is a vertex $v \in S_2^+ \setminus (S_1^+ \cup S^-)$ such that Q is composed of a (u, v)-dipath R_1 and a (d', v)-dipath R_2.

An ST_4-subdivision S is *special* if its d-vertex is d', its c-vertex is x_2, its a-vertex is not in $T \cup \{x_1, x_2\}$, and $x_1 \in V(S)$.

We check if D contains a special ST_4-subdivision. To do so, we check for every vertex a' in $V(D) \setminus (T \cup \{x_1, x_2\})$, if there are two independent $(\{x_1, x_2\}, a')$-dipaths Q_1 and Q_2 in D and an (a', S^+)-dipath R in $D - \{x_1, x_2\}$. If we find a vertex $a' \in V(D) \setminus (T \cup \{x_1, x_2\})$ such that three such dipaths exist, we return 'yes'. This is valid by the following claim.

Claim 40.5. *Let $a' \in V(D) \setminus (T \cup \{x_1, x_2\})$. If there are two independent $(\{x_1, x_2\}, a')$-dipaths Q_1 and Q_2 in D and an (a', S^+)-dipath R in $D - \{x_1, x_2\}$, then D contains an ST_4-subdivision.*

Subproof. The vertex $t(R)$ is in $S_2^+ \setminus S_1^+$ because $a' \notin T_1^-$. Thus, there is a $(t(R), x_2)$-dipath R_1 with internal vertices in S^+. Let y_2 be an outneighbour of x_2 in S^-. Since $a' \notin S^-$, the vertex y_2 is not on R_1. By definition of S^+ and S^-, there is a (y_2, R_1)-dipath R_2 in $D[S^+ \cup S^-]$.

Let C' be the directed cycle $x_2y_2 \cup R_2 \cup R_1[t(R_2), x_2]$. Since $y_2 \in S^-$, there is a directed (y_2, x_1)-dipath R_3 in $D[S^- \cup S^+ \cup \{x_1\}]$. This dipath does not intersect R_1 because $V(R_1) \setminus \{x_2\} \subseteq S_2^+ \setminus S_1^+$. Let z_2 be the last vertex along R_3 that lies in C'. The three vertices $x_2, z_2, t(R_2)$ are distinct. Moreover, the two dipaths Q_1 and Q_2 do not intersect C' for otherwise there would be a (y_2, a')-dipath in $D - \{x_1, x_2\}$ and a' would be in T^+. Thus $R_3 \cup Q_1$ contains a (y_3, a')-dipath R_3^* which is a (C', a')-dipath. Hence, we have two (C', a')-dipaths R_3^* and Q_2 and the (a', C')-dipath R' contained in $R \cup R_1[s(R_1), t(R_2)]$ whose vertices $s(R_3^*)$, $s(Q_2) = x_2$ and $t(R')$ on C' are pairwise distinct. Hence, by Lemma 38, D contains an ST_4-subdivision. $\Diamond$

If for every a' in $V(D) \setminus (T \cup \{x_1, x_2\})$, three dipaths Q_1, Q_2, R as used above do not exist, then D has no special ST_4-subdivision, which we will assume henceforth. Let D' be the digraph obtained from $D[T \cup \{x_1, x_2\}]$ by adding the arcs x_1x_2 and x_2x_1 if they were not in $A(D)$. Observe that D' has fewer vertices than D, because the vertices of $V(C) \setminus \{x_1, x_2\}$ are not in $V(D')$. We then return ST_4-`Subdivision`(D', d'). The validity of this recursive call is established by the following claim.

Claim 40.6. *D contains an ST_4-subdivision with d-vertex d' if and only if D' does.*

Subproof. From every ST_4-subdivision S' with d-vertex d' in D', one can obtain an ST_4-subdivision with d-vertex d' in D by replacing the arc x_1x_2 (resp. x_2x_1) by $C[x_1, x_2]$ (resp. $C[x_2, x_1]$).

Assume now that D contains an ST_4-subdivision S with d-vertex d'. Let a', b', and c' be the vertices in S corresponding to a, b, and c, respectively. Each arc in ST_4 corresponds to a dipath in S. We will denote these dipaths by $S[a', b']$, $S[b', c']$, etc.

Observe that in S, there are three internally disjoint directed paths (in both directions) between b' and d'. So $b' \in V(D')$, because directed paths between $V(D) \setminus V(D')$ and d' must go through $\{x_1, x_2\}$ by Property (T_2).

Next, we claim that $a' \in V(D')$. Suppose for a contradiction that $a' \notin V(D')$. Then both paths $S[d', a']$ and $S[c', a'] \cup S[c', d']$ must go through $\{x_1, x_2\}$ by Property (T_2). The path $S[a', b'] \cup S[d', b']$ is thus disjoint from $\{x_1, x_2\}$, and by (T_2) we have that $b' \in S_2^+ \setminus (S_1^+ \cup S^-)$. The path $S[b', c'] \cup S[c', d']$ must go through x_2 since $b' \notin S^- \cup S_1^+$. Thus, x_2 lies on $S[c', d']$. Since there is no special ST_4-subdivision in D, $c' \neq x_2$. Hence, $S[c', a']$ does not meet $\{x_1, x_2\}$, and the path $S[d', b'] \cup S[b', c'] \cup S[c', a']$ shows that $a' \in S^+$, a contradiction.

Let us prove that $c' \in V(D')$. Suppose for a contradiction that $c' \notin V(D')$. Then $c' \notin \{x_1, x_2\}$ and both, $S[d', b'] \cup S[b', c']$ and $S[c', d']$ must go through $\{x_1, x_2\}$. Moreover, x_2 is in $S[c', d']$ because x_1 has no outneighbour in S^-. Since x_2 is also on $S[d', a'] \cup S[a', b'] \cup S[b', c']$, we conclude that $x_2 \in S[b', c']$. Now, the path $S[d', a'] \cup S[c', a']$ gives a contradiction to the property (T_2).

We have shown that $\{a', b', c', d'\} \subseteq V(D')$. Therefore, the part of S outside D' is a directed path P whose initial vertex is dominated by $x \in \{x_1, x_2\}$ and whose terminal vertex dominates the vertex x' of $\{x_1, x_2\} \setminus \{x\}$. Hence, D' contains the ST_4-subdivision obtained from S by replacing xPx' by xx'. $\diamondsuit$

Each time we do a recursive call, the number of vertices decreases. So we do at most n of them. Between two recursive calls, we search for a W_2'-subdivision, which can be done in $O(n^3(n+m))$ by Theorem 35, and we compute some outsections and insection, which can be done in linear time. The only part that may need more time is in Case 2, when we check for every a' in $V(D) \setminus (T \cup \{x_1, x_2\})$ if D contains a special ST_4-subdivision. Each such test needs linear time by Claim 40.5. During this procedure, we either discover an ST_4-subdivision or not. If yes, we have spent at most $O(n(n+m))$ time for completing this task. Otherwise we spend linear time per vertex a', which is henceforth omitted when we proceed with the recursive call. This shows that ST_4-`Subdivision` runs in $O(n^4(n+m))$ time. $\square$

A.2 Others digraphs of order 4

There are also some digraphs with no big vertices and with order 4 that are not oriented graphs and that have not been proved tot be tractable without the Directed Grid Theorem. They are depicted in Figure 8. In this subsection, we give some elementary algorithms showing that those digraphs are tractable.

A.2.1 More tools

Some digraphs can be proved tractable using linkage, because k-LINKAGE is polymomial-time solvable when restricted to some classes of digraphs. This for example the case for acyclic digraphs, as shown

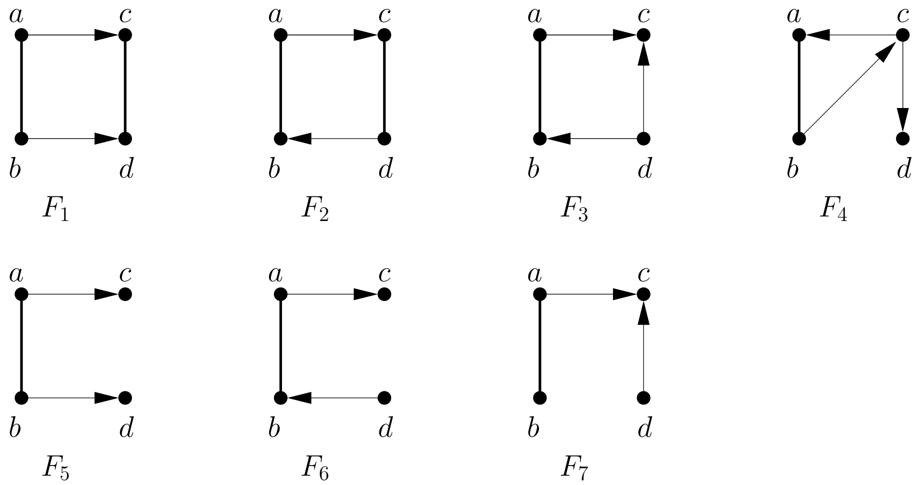


Figure 8: Some digraphs on 4-vertices, that are tractable. Bold undirected edges represent directed 2-cycles.

Fortune, Hopcroft and Wyllie [7].

Theorem 41 (Fortune, Hopcroft and Wyllie [7]). *For every fixed k the k-LINKAGE problem for acyclic digraphs can be solved in time $O(n^k(n+m))$.*

For every fixed k, k-LINKAGE is also polynomial-time solvable on digraphs of bounded DAG-width [3], on digraphs of bounded directed tree-width [9], or on tournaments [5].

A *feedback vertex set* or *cycle transversal* in a digraph D is a set of vertices S such that $D - S$ is acyclic. The minimum number of vertices in a cycle transversal of D is the *cycle-transversal number* and is denoted by $\tau(D)$. The maximum number of disjoint directed cycles in a digraph D is called the *cycle-packing number* and is denoted by $\nu(D)$. Clearly, $\nu(D) \leq \tau(D)$. Conversely, proving the so-called Gallai-Younger Conjecture, Reed et al. [12] proved that there exists a minimum function f such that $\tau(D) \leq f(\nu(D))$. It is obvious that $f(1) = 1$ and McCuaig [11] proved that $f(2) = 3$. Using this function f, Reed et al. [12] gave a polynomial-time algorithm to decide for every fixed k whether a digraph D contains k disjoint directed cycles. Basically, it tests all possible sets T of $f(k)$ vertices. If none of them is a cycle transversal, then it returns 'yes'. If one of them is a cycle transversal, it reduces the problem to a finite number (but depending on k) of $f(k)$-linkage problem in $D - T$. See also [8] for details.

Theorem 42 (Reed et al. [12]). *Let k be a fixed integer. There is an algorithm running in time $O(n^{f(k)}(n+m))$ that decides whether there are k disjoint directed cycles in a digraph.*

Theorem 42 and Lemma 5 directly imply the following.

Corollary 43. *Let k be a fixed integer. There is an algorithm running in time $O(n^{f(k)}(n+m)m)$ that finds k disjoint directed cycles in a digraph if they exist, and returns 'no' otherwise.*

In fact, Reed et al. proved the following stronger statement than Theorem 42.

Theorem 44 (Reed et al. [12]). *For any digraph F, F-SUBDIVISION is polynomial-time solvable when restricted to the class of digraphs with bounded cycle-transversal number.*

Note that this results is implied by the one of Berwanger et al. [3] stating that for every fixed k, k-LINKAGE is polynomial-time solvable on digraphs of bounded DAG-width.

A.2.2 F_1 is tractable

Theorem 45. F_1-SUBDIVISION *can be solved in* $O(n^4 m(n+m))$*-time.*

Proof. Let D be a digraph and let x be a vertex of D. An F_1-subdivision is x*-suitable*, if x is on the subdivision of the directed cycle aba.

We shall present a procedure F_1`-Subdivision`(D,x), that given a digraph D and a vertex x returns 'no' only if there is no x-suitable F_1-subdivision, and returns 'yes' when it finds an F_1-subdivision (not necessarily x-suitable). Moreover, this procedure runs in $O(n^3 m(n+m))$ time. Hence running F_1`-Subdivision`(D,x) for every vertex $x \in V(D)$, one solves F_1-SUBDIVISION in $O(n^4 m(n+m))$ time.

F_1`-Subdivision`(D,x) uses a subprocedure `Reduction`$(D,x,\mathcal{S})$ that, given a 1-separation $\mathcal{S} = (W_1, T, W_2)$ in D such that $x \in W_1$ and $W_2 \neq \emptyset$, reduces the problem to two smaller instances of F_1`-Subdivision`. `Reduction`$(D,x,\mathcal{S})$ proceeds as follows.

Let y be a vertex in W_2. We run a Menger algorithm that finds a 1-separation (W_1', T', W_2') of (x,y). The set W_1' is the set of vertices reachable from x in $D - T'$. We then replace $\mathcal{S}$ by $\mathcal{S}'$, that is, we set $W_1 := W_1'$, $T := T'$, and $W_2 := W_2'$. So now every vertex in W_1 can be reached from x.

If $T = \emptyset$, then we return F_1`-Subdivision`$(D[W_1], x)$. This is clearly valid since all the vertices of an x-suitable F_1-subdivision are in the outsection of x and thus cannot be in W_2 because there are no arcs from W_1 to W_2.

Suppose now that $|T| = 1$, say $T = \{t\}$. A vertex w_1 of W_1 is W_2*-reachable* if in D there exists a (t, w_1)-dipath whose internal vertices are all in W_2, and a vertex w_2 of W_2 is W_1*-reaching* if in D there exists a (w_2, t)-dipath whose internal vertices are all in W_1. Let D_1 be the digraph obtained from $D[W_1 \cup \{t\}]$ by adding the arc tw_1 (if it is not already in $A(D)$) for every W_2-reachable vertex $w_1 \in W_1$; let D_2 be the digraph obtained from $D[W_2 \cup \{t\}]$ by adding the arc $w_2 t$ (if it is not already in $A(D)$) for every W_1-reaching vertex w_2 of W_2.

`Reduction`$(D,x,\mathcal{S})$ returns (F_1`-Subdivision`(D_1, x) or F_1`-Subdivision`(D_2, t)).

The validity of the subprocedure `Reduction` is justified by the following claim.

Claim 45.1. (i) *If D contains an x-suitable F_1-subdivision, then either D_1 contains an x-suitable F_1-subdivision or D_2 contains a t-suitable F_1-subdivision.*

(ii) *For any $i = 1, 2$, if D_i contains an F_1-subdivision, then D contains an F_1-subdivision.*

Subproof. (i) Assume that D contains an x-suitable F_1-subdivision S. Let C_1 and C_2 be the directed cycles in S corresponding to aba and cdc, respectively, and let P_1 and P_2 be the two disjoint $(V(C_1), V(C_2))$-dipaths in S. By definition, $x \in V(C_1)$.

We distinguish several cases according to the position of C_1 and C_2.

Assume first that C_1 is contained in $D[W_1 \cup \{t\}]$. Since all dipaths from W_1 to W_2 go through t, one of the P_i, say P_1, is in $D[W_1]$, and $V(C_2) \cap W_1 \neq \emptyset$.

- Suppose that C_2 is in $D[W_1]$. If $V(P_2) \cap W_2 = \emptyset$, then S is an x-suitable F_1-subdivision in D_1. If $V(P_2) \cap W_2 \neq \emptyset$, then there is a vertex w_1 of $W_1 \cap V(P_2)$ such that $P_2 \cap D[W_2] = P_2]t, w_1[$. Therefore the digraph S_1 obtained from S by replacing $P_2[t, w_1]$ by the arc tw_1 is an x-suitable F_1-subdivision in D_1.

- Suppose now that $V(C_2) \cap W_2 \neq \emptyset$. Then necessarily $t \in V(C_2)$, and P_1 and P_2 are in $D[W_1 \cup \{t\}]$. Moreover, there is a vertex w_1 of $W_1 \cap V(C_2)$ such that $C_2 \cap D[W_2] = C_2]t, w_1[$. Therefore the digraph S_1 obtained from S by replacing $C_2[t, w_1]$ by the arc tw_1 is an x-suitable F_1-subdivision in D_1.

Suppose now that $V(C_1) \cap W_2 \neq \emptyset$. Since $x \in V(C_1)$, C_1 necessarily contains t, because there is no arcs from W_1 to W_2. Moreover, there exist two vertices $w_1 \in W_1 \cap V(C_1)$ and $w_2 \in W_2 \cap V(C_1)$ such that $C_1 = C_1[t, w_2]w_2w_1C_1[w_1, t]$, $D[W_1] \cap C_1 = C_1[w_1, t[$ and $D[W_2] \cap C_1 = C_1]t, w_2]$. Now, C_2 is contained in $D[W_1 \cup W_2]$, and so C_2 is contained either in $D[W_1]$ or in $D[W_2]$.

- Assume that C_2 is in $D[W_1]$. Let w_1^i be the first vertex along P_i in W_1. Since there is no arc form W_1 to W_2, all vertices in $P_i[s(P_i), w_1^i[$ are in W_2 and all vertices in $P_i[w_1^i, t(P_i)]$ are in W_1.

  a) If $s(P_1) = w_1^1$ and $s(P_2) = w_1^2$, then the digraph S_1 obtained from S by replacing $C_1[t, w_1]$ by the arc tw_1 is an x-suitable F_1-subdivision in D_1.

  b) If $s(P_1) \neq w_1^1$ and $s(P_2) = w_1^2$, then the digraph S_1 obtained from S by replacing $C_1[t, w_1]$ and $P_1[s(P_1), w_1^1]$ by the arcs tw_1 and tw_1^1 is an x-suitable F_1-subdivision in D_1.

  c) Assume finally that $s(P_1) \neq w_1^1$ and $s(P_2) \neq w_1^2$. Then both $s(P_1)$ and $s(P_2)$ are in $W_2 \cup \{t\}$. Since every vertex of W_1 is reachable from x in $D[W_1]$, there is a $(V(C_1) \cap W_1, V(C_2))$-dipath Q in $D[W_1]$. Observe that $s(Q)$ is distinct from $s(P_1)$ and $s(P_2)$ because it is in W_1. If Q does not intersect $P_1 \cup P_2$, set $Q' := Q$. Otherwise, without loss of generality, the first vertex z along Q in $P_1 \cup P_2$ is in P_2. In this case, set $Q' = Q[s(Q), z]P_2[z, t(P_2)]$. In both cases, the subdigraph S' obtained from S by replacing P_2 by Q' is an F_1-subdivision. Now Subcase (b) applies to S', so D_1 contains an x-suitable F_1-subdivision in D_1.

- Assume that C_2 is in $D[W_2]$. Then P_1 and P_2 are in $D[W_2 \cup \{t\}]$, and so $C_1 \cap D[W_1] = C_1[w_1, t[$. Hence the digraph S_2 obtained from S by replacing $C_1[w_2, t]$ by the arc w_2t is a t-suitable F_1-subdivision in D_2.

(ii) Suppose that S_1 is an F_1-subdivision in D_1. By construction of D_1, all arcs of $A(S_1) \setminus A(D)$ are joining t to some W_2-reachable vertex. Since each vertex in F_1 has outdegree at most 2, there are at most two arcs in $A(S_1) \setminus A(D)$.

If there is no in arc in $A(S_1) \setminus A(D)$, then S_1 is an F_1-subdivision in D. If there is a unique arc tw_1 in $A(S_1) \setminus A(D)$, then the digraph S obtained from S_1 by replacing the arc tw_1 by a (t, w_1)-dipath with internal vertices in W_2 is an F_1-subdivision contained in D. Assume finally that $A(S_1) \setminus A(D)$ contains two arcs, tw_1 and tw_1'. Note that t has indegree 1 and outdegree 2 in S_1. Let P (resp. P') be a (t, w_1)-dipath (resp. (t, w_1')-dipath) with all internal vertices in W_2. Let t' be the last vertex along P' which is in $V(P) \cap V(P')$. Now the digraph S obtained from S_1 by replacing tw_1 and tw_1' by the union of P and $P'[t', w_1']$ is an F_1-subdivision contained in D.

A similar argument shows that if D_2 contains an F_1-subdivision, then D contains an F_1-subdivision. $\Diamond$

Using `Reduction`, we construct another procedure `cleaning`(C_1, C_2, x) that given two disjoint directed cycles C_1 and C_2 and the vertex x, either reduces the problem or finds a pair of disjoint directed cycles (C_1', C_2') such that $x \in V(C_1')$. This procedure proceeds as follows.

If C_1 contains x, then we set $(C_1', C_2') := (C_1, C_2)$. If C_2 contains x, then we set $(C_1', C_2') := (C_2, C_1)$.

Assume now that x is not in $V(C_1 \cup C_2)$. We first check whether there is a cycle C containing x. If not, then we return 'no' because D does certainly not contain any x-suitable F_1-subdivision. If C does not intersect C_1, then we set $(C_1', C_2') := (C, C_1)$. If C does not intersect C_2, then we set $(C_1', C_2') := (C, C_2)$. Henceforth, C intersects both C_1 and C_2. Let y be the first vertex after x in C that is in $V(C_1 \cup C_2)$, and let z be the last vertex before x in C that is in $V(C_1) \cup V(C_2)$. Free to permute the indices of C_1 and C_2, we may assume that $y \in V(C_1)$. Moreover, if $z \in V(C_1)$, then we set $(C_1', C_2') := (C[x, y]C_1[y, z]C[z, x], C_2)$. So we may assume that $z \in V(C_2)$. Using a Menger algorithm, we check whether there are two disjoint (x, C_2)-dipaths. If not, then we obtain a 1-separation $\mathcal{S} = (W_1, T, W_2)$ in D such that $x \in W_1$ and $V(C_2) \subseteq T \cup W_2$. In that case, we return `Reduction`$(D, x, \mathcal{S})$. Suppose now that there are two independent (x, C_2)-dipaths Q_1 and Q_2. If Q_i does not intersect C_1, then the closed walk $Q_i C_2[t(Q_i), y]C[y, x]$ contains a cycle through x. We return this cycle and C_1 as (C'_1, C'_2). If Q_1 and Q_2 both intersect C_1, then there are two disjoint (C_1, C_2)-dipaths, whose union with C_1 and C_2 is an F_1-subdivision. So we return 'yes'. This finishes the subprocedure `cleaning`(C_1, C_2, x).

Finally, let us describe F_1`-Subdivision`(D, x).

We first check whether there are two disjoint directed cycles in D. If not, then we return 'no' because D cannot contain an F_1-subdivision in this case. Henceforth, we may assume that there are two disjoint directed cycles Γ_1 and Γ_2.

We then run `cleaning`(Γ_1, Γ_2, x). If the instance was not reduced by this procedure, we get two disjoint directed cycles (Γ'_1, Γ'_2) such that $x \in V(\Gamma'_1)$.

We run a Menger algorithm to check whether there are two disjoint $(V(\Gamma'_1), V(\Gamma'_2))$-dipaths. If two such dipaths P_1 and P_2 exist, then $\Gamma'_1 \cup \Gamma'_2 \cup P_1 \cup P_2$ is an x-suitable F_1-subdivision, and we return 'yes'. If not, then the Menger algorithm returns a 1-separation $\mathcal{S} = (W_1, T, W_2)$ of $(V(\Gamma'_1), V(\Gamma'_2))$. If $x \in W_1$, then we return `Reduction`$(D, x, \mathcal{S})$. If $x \notin W_1$, then $T = \{x\}$. In this case we proceed as described below.

From now on, we may assume that D contains no x-suitable F_1-subdivision in which the cycle through x lies in $D[W_2 \cup \{x\}]$ and the other cycle lies in $D[W_1]$. This is guaranteed either by nonexistence of C_1 or C_2, or by the outcome of the previous step.

Let D'_1 be the digraph obtained from D by contracting W_2 into a vertex u and let D_2 be the digraph obtained from $D[W_2 \cup \{x\}]$ by adding the arc $w_2 x$ (if it is not already in $A(D)$) for every W_1-reaching vertex w_2 of W_2. We return (F_1`-Subdivision`(D'_1, x) or F_1`-Subdivision`(D_2, x)).

This is valid by the following claim whose proof is very similar to the one of Claim 45.1.

Claim 45.2. (i) *If D contains an x-suitable F_1-subdivision, then either D'_1 contains an x-suitable F_1-subdivision or D_2 contains a x-suitable F_1-subdivision.*

(ii) *If D'_1 or D_2 contains an F_1-subdivision, then D contains an F_1-subdivision.*

Subproof. (i) Assume that D contains an x-suitable F_1-subdivision S. Let C_1 and C_2 be the directed cycles in S corresponding to aba and cdc, respectively, and let P_1 and P_2 be the two disjoint $(V(C_1), V(C_2))$-dipaths in S. By definition, $x \in V(C_1)$.

We distinguish several cases according to the positions of C_1 and C_2. Suppose first that C_1 is contained in $D[W_1 \cup \{x\}]$. Since all dipaths from W_1 to W_2 go through x, one of the P_i, say P_1, is in $D[W_1]$, and $V(C_2) \cap W_1 \neq \emptyset$. Hence C_2 is in $D[W_1]$. If $V(P_2) \cap W_2 = \emptyset$, then S is an x-suitable F_1-subdivision in D'_1. If $V(P_2) \cap W_2 \neq \emptyset$, then P_2 contains x and there is a vertex w_1 of $W_1 \cap V(P_2)$ such that $P_2 \cap D[W_2] = P_2]x, w_1[$. Therefore the digraph S_1 obtained from S by replacing $P_2[x, w_1]$ by the dipath xuw_1 is an x-suitable F_1-subdivision in D'_1.

The second possibility is that $C_1 \subseteq D[W_2 \cup \{x\}]$. Since C_1 contains x, C_2 is contained in $D[W_1 \cup W_2]$, and so C_2 is contained either in $D[W_1]$ or in $D[W_2]$. If C_2 lies in $D[W_1]$, then the digraph S' obtained from S by contracting all the vertices of W_2 into U is an x-suitable F_1-subdivision in D'_1. If $C_2 \subseteq D[W_2]$, then S is an x-suitable F_1-subdivision in D_2.

Finally, suppose that $V(C_1) \cap W_2 \neq \emptyset$ and $V(C_1) \cap W_1 \neq \emptyset$. Then there exist two vertices $w_1 \in W_1 \cap V(C_1)$ and $w_2 \in W_2 \cap V(C_1)$ such that $C_1 = C_1[x, w_2]w_2 w_1 C_1[w_1, x]$, $D[W_1] \cap C_1 = C_1[w_1, x[$, and $D[W_2] \cap C_1 = C_1]x, w_2]$. Now C_2 is contained in $D[W_1 \cup W_2]$, and so C_2 is contained either in $D[W_1]$ or in $D[W_2]$.

- Suppose first that $C_2 \subseteq D[W_1]$. Let w_1^i be the first vertex along P_i in W_1. Since there is no arc form W_1 to W_2, all vertices of $P_i[s(P_i), w_1^i[$ are in W_2 and all vertices of $P_i[w_1^i, t(P_i)]$ are in W_1.

a) If both P_1 and P_2 are contained in $D[W_1 \cup \{x\}]$, then the digraph S_1 obtained from S by replacing $C_1[x, w_1]$ by the dipath xuw_1 is an x-suitable F_1-subdivision in D'_1.

b) If one of the two P_i's, say P_1 is contained in $D[W_1 \cup \{x\}]$, then the digraph S_1 obtained from S by replacing $C_1[x, w_1]$ and $P_1[s(P_1), w_1^1]$ by the dipaths xuw_1 and uw_1^1) is an x-suitable F_1-subdivision in D'_1.

c) Assume now that P_1 and P_2 both intersect W_2. Both $s(P_1)$ and $s(P_2)$ are in $W_2 \cup \{x\}$. Observe that every vertex of W_1 is reachable from $\Gamma'_1 \setminus \{x\}$ in D_1, so there is an (x, C_2)-dipath Q in $D[W_1 \cup \{x\}]$. If Q does not intersect $P_1 \cup P_2$, then set $Q' := Q$. Otherwise, we may assume that the first vertex z in $V(P_1 \cup P_2)$ along $Q - x$ in P_2. Set $Q' := Q[x, z]P_2[z, t(P_2)]$. Observe that Q' and P_1 are disjoint except possibly in x. Now the digraph S_1 obtained from $C_1 \cup C_2 \cup P_1 \cup Q'$ by replacing $C_1[x, w_1]$ and $P_1[s(P_1), w_1^1]$ by the dipaths xuw_1 and uw_1^1 is an x-suitable F_1-subdivision in D'_1.

- The second possibility is that $C_2 \subseteq D[W_2]$. Then P_1 and P_2 are in $D[W_2 \cup \{x\}]$, and so $C_1 \cap D[W_1] = C_1[w_1, x[$. Hence the digraph S_2 obtained from S by replacing $C_1[w_2, x]$ by the arc w_2x is an x-suitable F_1-subdivision in D_2.

(ii) We already showed in Claim 45.1 that if D_2 contains an F_1-subdivision, then D contains an F_1-subdivision. Let us now prove that if D'_1 contains an F_1-subdivision, then D contains an F_1-subdivision.

Suppose that S_1 is an F_1-subdivision in D'_1. If $u \notin V(S_1)$, then S_1 is an F_1-subdivision in D. If $u \in V(S_1)$, then S_1 contains a dipath (x, u, w_1) for some $w_1 \in W_1$, and possibly one other arc uw'_1. By definition of D'_1, there are a (t, w_1)-dipath P and a (t, w'_1)-dipath P' with internal vertices in W_2. Let t' be the last vertex along P' which is in $V(P) \cap V(P')$. Then the digraph S obtained from S_1 by replacing tw_1 by P and tw'_1 (if it exists) by $P'[t', w'_1]$ is an F_1-subdivision contained in D. $\Diamond$

Let us now estimate the complexity of F_1-$\texttt{Subdivision}(D, x)$. This procedure first finds two disjoint directed cycles and then runs a few Menger algorithms and either returns an answer or make a recursive call on two smaller instances, which are either D_1 and D_2, or D'_1 and D_2. Two disjoint directed cycles can be found in $O(n^3m(n + m)$ by Corollary 43.

The smaller instances D_1 and D'_1 can be constructed in linear time: indeed a vertex $w_1 \in W_1$ is W_2-reachable if and only if it is has an inneighbour in the outsection of x in $D[W_2 \cup \{x\}]$, and so all W_2-reachable vertices can be found in linear time. Similarly, the set of W_1-reaching vertices in W_2 can be determined in linear time, and thus D_2 can be constructed in linear time. Hence F_1-$\texttt{Subdivision}(D, x)$ makes at most cn^2 operations before calling recursively, for some absolute constant c.

Let us denote by $T(n)$ the maximum time for F_1-$\texttt{Subdivision}(D, x)$ on a digraph with n vertices. Since $|V(D_1)| + |V(D_2)| = |V(D)| + 1$ and $|V(D'_1)| + |V(D_2)| = |V(D)| + 2$, we have

$$T(n) \leq \max\{T(n_1) + T(n_2) + cn^2 \mid n_1, n_2 < n \text{ and } n_1 + n_2 \in \{n + 1, n + 2\}\}.$$

This implies that $T(n) \leq O(n^3)$.

Therefore F_1-$\texttt{Subdivision}$ runs in time $O(n^3m(n + m))$. $\square$

A.2.3 F_2 is tractable

The aim of this subsubsection is to prove that the digraph F_2depicted on Figure 8 is tractable.

Theorem 46. F_2-SUBDIVISION *can be solved in* $O(n^4m(n + m))$ *time.*

In order to prove Theorem 46, we need some preliminary results.

Let F be a subdigraph of a digraph D. An *ear* of F in D is an oriented path in D containing at least one edge, whose endvertices lie in F but whose edges and internal vertices do not belong to F. A *directed ear* of F is an ear of F that is a directed path. A digraph is said to be *robust* if it is strong and 2-connected. The following lemma is well-known; it is very similar to Proposition 5.11 of [4].

Lemma 47. *Let F be a non-trivial strong subdigraph of a robust digraph D. Then F has a directed ear in D. Moreover such a directed ear can be found in time $O(n(n+m))$.*

Proof. Because D is 2-connected, F has an oriented ear in D. Among all such ears, we choose one in which the number of reverse arcs (those directed towards its initial vertex) is as small as possible. We show that this path P is in fact a directed ear.

Assume the contrary, and let uv be a backward arc of P. Because D is strong, there exist in D an (F,u)-dipath Q and a (v,F)-dipath R (one of which might be of length zero). The initial vertex of Q and the terminal vertex of R must be one and the same vertex, for otherwise the directed walk $QuvR$ would contain a directed ear of F, contradicting the choice of P and our assumption that P is not a directed ear. Let this common vertex be z. We may assume that $z \neq s(P)$ (the case $z \neq t(P)$ being analogous). Then the $(s(P),z)$-walk $P[s(P),v]Rz$ contains an oriented $(s(P),z)$-path that contradicts the choice of P. Thus P is indeed a directed ear of F.

A directed ear of F may be found by running a search from each vertex of F in $D \setminus A(F)$. Hence it can be found in $O(n(n+m))$ steps. □

Let D_1 and D_2 be two subdigraphs in D. Two dipaths are (D_1,D_2)-*opposite* if they are disjoint and one of them is a (D_1,D_2)-dipath and the other is a (D_2,D_1)-dipath. Opposite dipaths play an important role in detecting F_2-subdivisions because of the following easy lemma.

Lemma 48. *Let D be a digraph and D_1 and D_2 two disjoint non-trivial strong subdigraphs of D. If there are (D_1,D_2)-opposite dipaths in D, then D contains an F_2-subdivision.*

Proof. Let P_1 and P_2 be two (D_1,D_2)-opposite dipaths, with P_1 a (D_1,D_2)-dipath and P_2 a (D_2,D_1)-dipath. Since D_i is strong, there is an $(t(P_{3-i}), s(P_i))$-dipath Q_i in D_i. For the same reason, there is a $(Q_i - s(Q_i), s(Q_i))$-dipath R_i in D_i. Now $P_1 \cup P_2 \cup Q_1 \cup Q_2 \cup R_1 \cup R_2$ is an F_2-subdivision in D. □

Proof of Theorem 46. We shall present a recursive procedure F_2-`Subdivision`(D), that given a digraph D decides whether it contains an F_2-subdivision or not.

This procedure proceeds as follows. We first check whether D is robust. If not, then we solve the problem for each robust component separately. Henceforth, we may assume that D is robust.

We next check whether there are two disjoint directed cycles. If not, then we return 'no' since F_2 contains two disjoint directed cycles. If two such cycles C_1 and C_2 exist, then we compute the strong component D_1 of C_1 in $D - C_2$, and next the strong component D_2 of C_2 in $D - D_1$. Hence D_1 and D_2 are two disjoint non-trivial strong subdigraphs in D. Moreover, they satisfy the following property.

Claim 48.1. *If P is a (D_1,D_2)-dipath and Q is a (D_2,D_1)-dipath, then P and Q are internally disjoint.*

Subproof. Suppose, by way of contradiction, that P and Q have a common internal vertex x. The vertex x is in the strong component of D_1 in $D - D_2$. Hence it is in the strong component of C_1 in $D - C_2$. So $x \in D_1$, a contradiction. ◊

We check whether there are (D_1,D_2)-opposite dipaths in D. By using Claim 48.1 this task reduces to finding a (D_1,D_2)-dipath and a (D_2,D_1)-dipath whose endvertices are disjoint. If there are such paths, then, by Lemma 48, D contains an F_2-subdivision and we return 'yes'. Henceforth we may assume that there are no (D_1,D_2)-opposite dipaths in D.

By Lemma 47, there is a directed ear P_1 of D_1. Since $D_1 \cup P_1$ is strong, P_1 must intersect D_2. Furthermore, the intersection of P_1 and D_2 is reduced to a single vertex, because there are no (D_1,D_2)-opposite dipaths. Let u_1 be the initial vertex of P_1, v_1 the terminal vertex of P_1, and let u_2 be the vertex of $P_1 \cap D_2$. By Lemma 47, there is a directed ear P_2 of D_2. If the terminal vertex of P_2 is u_2, then we consider the converse of D, D_1, D_2, P_1 and P_2. (This is valid since F_2 is its own converse.) Hence, we may assume that the terminal vertex v_2 of P_2 is different from u_2. Similarly to P_1, the directed ear P_2 intersects D_1 in a single vertex w_1. Necessarily, $w_1 = v_1$ for otherwise $P_1[u_2, v_1]$ and $P_2[w_1, v_2]$ are (D_1, D_2)-opposite, by Claim 48.1. Also, the initial vertex of P_2 is u_2, and thus we may assume that both ears have common segment $P_1[u_2, v_1] = P_2[u_2, v_1]$. Furthermore, $P_1[u_1, u_2]$ and $P_2[v_1, v_2]$ are disjoint for otherwise, there are two (D_1, D_2)-opposite dipaths.

Set $P = P_1P_2[v_1, v_2]$. We check whether $D - (D_1 \cup D_2)$ contains a non-trivial strong component D_3. If D_3 exists and intersects both $P[u_1, v_1]$ and $P[u_2, v_2]$, then we return 'yes'. This is valid by the following claim.

Claim 48.2. *If D_3 intersects both $P[u_1, v_1]$ and $P[u_2, v_2]$, then D contains an F_2-subdivision.*

Subproof. Suppose first that D_3 intersects $P[u_2, v_1]$. If D_3 also intersects $P[u_1, u_2]$, then there are two (D_1, D_3)-opposite dipaths, and so by Lemma 48, D contains an F_2-subdivision. Similarly, if D_3 also intersects $P[v_1, v_2]$, then there are two (D_2, D_3)-opposite dipaths, and so D contains an F_2-subdivision. Hence we may assume that D_3 does not intersect $P[u_1, u_2] \cup P[v_1, v_2]$. If D_3 intersects $P[u_2, v_1]$ in more than one vertex, then let u_3 (resp. v_3) be the first (resp. last) vertex of D_3 along $P[u_2, v_1]$. The two dipaths $P[u_1, u_3]$ and $P[v_3, v_1]$ are (D_1, D_3)-opposite. Hence, by Lemma 48, D contains an F_2-subdivision.

So D_3 intersects $P[u_2, v_1]$ in a unique vertex, say w_3. By Lemma 47, there is a directed ear P_3 of D_3. By definition of D_1 and D_2, P_3 intersects both D_1 and D_2. Now one of the two endvertices of P_3, say u_3, is distinct from w_3.

If u_3 is the initial vertex of P_3, then consider the first vertex v_3 along P_3 in $D_1 \cup D_2$. By definition of D_2, $v_3 \in V(D_1)$. Now $P_3[u_3, v_3]$ is disjoint from $P[u_2, w_3]$ since D_3 is a strong component of $D - D_1$. Furthermore $P_3[u_3, v_3]$ is internally disjoint from $P[u_1, u_2]$ because the dipaths $P[u_1, u_2]$ and $P[u_2, w_3]D_3[w_3, u_3]P_3[u_3, v_3]$ are internally disjoint by Claim 48.1. If $v_3 \neq u_1$, then $P[u_1, w_3]$ and $P_3[u_3, v_3]$ are (D_1, D_3)-opposite, and so by Lemma 48, D contains an F_2-subdivision. Finally, if $v_3 = u_1$, then $P[v_1, v_2]$ is disjoint from $P_3[u_3, v_3]$. In this case, $P_3[u_3, v_3]$ and $P[v_1, v_2]D_2[v_2, u_2]P[u_2, w_3]$ are (D_1, D_3)-opposite paths giving an F_2-subdivision.

If u_3 is the terminal vertex of P_3, then we get the result analogously.

Suppose now that D_3 does not intersect $P[u_2, v_1]$. Then it must intersect both $P[u_1, u_2]$ and $P[v_1, v_2]$. Let u_3 be the first vertex of D_3 along $P[u_1, u_2]$ and v_3 be the last vertex of D_3 along $P[v_1, v_2]$. Now $P[u_1, u_3]$ and $P[v_3, v_2]D_2[v_2, u_2]P[u_2, v_1]$ are two (D_1, D_3)-opposite dipaths. Thus, by Lemma 48, D contains an F_2-subdivision. $\lozenge$

If D_3 exists, we are either done by Claim 48.2, or D_3 is disjoint from one of the paths, $P[u_1, v_1]$ or $P[u_2, v_2]$. Now, we replace D_1 by the strong component of $D - D_3$ containing $D_1 \cup D_2$. Observe that this makes the order of D_1 increase. Further, we replace D_2 by D_3 and replace C_2 by a cycle in this new strong digraph. By doing this, Claim 48.1 remains valid. By repeating the process as long as possible, we reach the situation where all strong components of $D - (D_1 \cup D_2)$ are trivial, that is $D - (D_1 \cup D_2)$ is acyclic. We check whether $D_2 - u_2$ contains a directed cycle. If it contains such a cycle C_2', then let D_1' be the strong component of C_1 in $D_2 - C_2'$ and D_2' the strong component of C_2' in $D - D_1'$. Clearly, D_1' is a superdigraph of $D_1 \cup P_1$, so $|D_1'| > |D_1|$. Hence, we replace D_1, D_2 by D_1' and D_2', respectively, and repeat the procedure for the new pair D_1, D_2. So we may assume that there is no cycle in $D_2 - u_2$.

Moreover, if there is a (v_2, u_1)-dipath Q whose internal vertices are not in $V(D_1 \cup D_2)$, we also check whether there is a cycle in $D_2 - v_2$. If yes, then as above we find new pair of non-trivial strong digraphs (D_1', D_2') with $|D_1'| > |D_1|$. Hence in that case, we may also assume that there is no cycle in $D_2 - v_2$.

Let D^* be the digraph obtained from D by contracting D_2 into a single vertex w^*. We return F_2-`Subdivision`(D^*). The following claim shows that this recursive call is valid.

Claim 48.3. *D contains an F_2-subdivision if and only if D^* contains an F_2-subdivision.*

Subproof. Suppose that D contains an F_2-subdivision S. Let C_1 and C_2 be the two disjoint directed cycles in S corresponding to the subdivision of aba and cdc. Observe that each C_i intersects $D_1 \cup D_2$, because there is no strong component in $D - (D_1 \cup D_2)$.

C_1 and C_2 cannot be in both in D_2 for otherwise one of the two avoids u_2, which is impossible. Moreover, one of the cycles cannot be in D_1 while the other one is in D_2 for otherwise the (C_1, C_2)- and (C_2, C_1)-dipaths in S would contains two (D_1, D_2)-opposite dipaths in D, which is impossible. If C_1 and C_2 are both contained in D_1, then either S is contained in D_1, in which case it is also in D^*, or the arcs of S which are not in $A(D_1)$ induce a directed ear R which intersects with D_2 in a single vertex w_2, because there are no (D_1, D_2)-opposite dipaths. Hence the digraph S^* obtained from D by replacing the vertex w_2 by w^* is an F_2-subdivision in D^*. So we may assume that one of the cycles intersects D_1 and D_2.

<u>Case 1:</u> There is no (v_2, u_1)-dipath whose internal vertices are not in $V(D_1 \cup D_2)$. In that case, all (D_2, D_1)-dipaths are (u_2, v_1)-dipaths. Therefore, the two cycles C_1 and C_2 cannot both intersect D_1 and D_2. Thus one of them, say C_1, does not intersect both, and thus must be contained in D_1. Consequently, C_2 intersects both D_1 and D_2. Thus C_2 contains a (u_2, v_1)-dipath. Therefore, the (C_2, C_1)-dipath in S must be in D_1, and the (C_1, C_2)-dipath intersects D_2 in u_2. Therefore the digraph S^* obtained from S by contracting the vertices of $V(S) \cap V(D_2)$ into w^* is an F_2-subdivision in D^*.

<u>Case 2:</u> There is a (v_2, u_1)-dipath Q whose internal vertices are not in $V(D_1 \cup D_2)$. In this case, all (D_1, D_2)-dipaths are (u_1, u_2)- or (v_1, v_2)-dipaths because there are no (D_1, D_2)-opposite dipaths in D. For the same reason, all (D_2, D_1)-dipaths are (u_2, v_1)- or (v_2, u_1)-dipaths. Therefore, the two cycles C_1 and C_2 cannot both intersect D_1 and D_2. Thus one of them, say C_1, does not intersect both, and thus must be contained in D_1. Consequently, C_2 intersects both D_1 and D_2.

We are in one of the three following cases: $C_2 = P[u_1, v_1]D_1[v_1, u_1]$, $C_2 = P[u_2, v_2]D_2[v_2, u_2]$, or $C = PQ$. In each of these cases, one can see that the digraph S^* obtained from S by contracting the vertices of $V(S) \cap V(D_2)$ in w^* contains an F_2-subdivision in D^*.

Conversely, suppose that D^* contains an F_2-subdivision S^*. If S^* does not contains w^*, then it is contained in D. So we may assume that S^* contains w^*.

Suppose w^* has indegree and outdegree 1 in S^*. Let u (resp. v) be the inneighbour (resp. outneighbour) of w^* in S^*. By definition of D^*, the vertex u has an outneighbour u_2' in D_2 and the vertex v has an inneighbour v_2' in D_2. Hence the digraph S obtained from S^* by replacing the dipath uw^*v by the dipath $uD[u_2', v_2']v$ is an F_2-subdivision in D.

Suppose w^* has indegree 1 and outdegree 2 in S^*. Let u be the inneighbour of w^* in S^* and let v and v' be the outneighbours of w^* in S. By definition of D^*, the vertex u has an outneighbour u_2' in D_2 and the vertex v (resp. v') has an inneighbour v_2' (resp. v_2'') in D_2. Let P be a (u_2', v_2')-dipath in D_2 and Q be a (P, v_2'')-dipath in D. The digraph S obtained from S^* by replacing the vertex w^* by $P \cup Q$ is an F_2-subdivision in D.

If w^* has indegree 2 and outdegree 1, we find an F_2-subdivision in D in a similar way. $\Diamond$

Let us now estimate the time complexity of F_2-`Subdivision`. The procedure first constructs the digraphs D_1 and D_2. It requires to find two disjoint directed cycles and then to compute two strong components. By Corollary 43 this can be done in time $O(n^3m(n+m))$. Next, the algorithm checks a few times for opposite paths, and for directed cycles, before either increasing the order of D_1 or making a recursive call. Checking if there are (D_1, D_2)-opposite paths can be done in $O(n(n+m))$ time by running searches in $D \setminus A(D_1 \cup D_2)$ from each vertex, and finding if there is a directed cycle in a digraph can be done in $O(n(n+m))$ time by checking for each vertex v if there is a (v, v)-handle. Thus, since the order of D_1 increases at most $O(n)$ times, there are at most $O(nm(n+m))$ such operations between two recursive calls. Hence the time between two recursive calls is at most $O(n^3m(n+m))$. At each call, the order of the instance digraph decreases. Hence the time complexity of F_2-`Subdivision` is $O(n^4 m(n+m))$. $\square$

A.2.4 F_3 is tractable

Theorem 49. *F_3 is tractable.*

Given two vertices c' and d' in D, a (c', d')*-forced* F_3-subdivision is an F_3-subdivision in D with c-vertex c' and d-vertex d'.

To prove Theorem 49, we shall describe a polynomial-time algorithm to solve F_3-SUBDIVISION. The key ingredient of our algorithm is a polynomial-time procedure F_3-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$ whose input is a strong digraph D and two sets of two vertices $\{c_1, c_2\}, \{d_1, d_2\}$ ($c_1 \neq c_2$ and $d_1 \neq d_2$). Let $\hat{D}(\{c_1, c_2\}, \{d_1, d_2\})$ be the digraph obtained from D by adding two new vertices c'', d'' and the four arcs $c_1c'', c_2c'', d''d_1, d''d_2$. The procedure F_3-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$ returns 'yes' if it finds an F_3-subdivision in $\hat{D}$, and returns 'no' if $\hat{D}$ has no (c'', d'')-forced F_3-subdivision.

Before describing the procedure F_3-`Strong+`, let us describe the algorithm for F_3-SUBDIVISION, assuming we have such a procedure.

The algorithm For every vertex c' and d', we run a procedure F_3-`Forced`(c', d') that returns 'yes' if it finds an F_3-subdivision, and return 'no' if it finds evidence that there is no (c', d')*-forced* F_3-subdivision in D. Since there are $O(n^2)$ possible choices of c' and d', if F_3-`Forced` runs in polynomial time, the overall algorithm will also run in polynomial time.

F_3-`Forced`(c', d') proceeds as follows. We first compute the strong components $G_1, \dots, G_p$ of $D - \{c', d'\}$. Observe that the directed cycle (corresponding to aba) in an F_3-subdivision must be contained in one of the strong components. For each strong component G_i, we run a procedure F_3-`Suitable`(c', d', G_i) that returns 'yes' if it finds an F_3-subdivision and returns 'no' only if there is no (c', d')-forced F_3-subdivision whose directed cycle is in G_i. Such a subdivision is called (c', d', G_i)-suitable.

We first test if there is a (d', c')-dipath in $D - G_i$. We then run two separate procedures depending on whether or not such a path exists.

<u>Case 1:</u> Assume there is a (d', c')-dipath P in $D - G_i$. Let X be the set of vertices $x \in V(G_i)$ that are terminal vertices of a (d', G_i)-dipath in D. The set X can be computed in linear time by running a search from d' in the digraph obtained from D by deleting all the arcs having their tail in G_i. Let Y be the set of vertices $y \in V(G_i)$ that are initial vertices of a (G_i, c')-dipath in D. Similarly to X, the set Y can be determined in linear time.

If there are no two distinct vertices $x \in X$ and $y \in Y$, then we return 'no'. Otherwise we return 'yes'. This is valid according to the following claim.

Claim 49.1. (i) *If there are no two distinct vertices $x \in X$ and $y \in Y$, then D contains no (c', d', G_i)-suitable F_3-subdivision.*

(ii) *If there are two distinct vertices $x \in X$ and $y \in Y$, then D contains an F_3-subdivision.*

Subproof. (i) If D contains a (c', d', G_i)-suitable F_3-subdivision S, then consider the directed cycle C in S. This cycle is in G_i. Moreover, in S, there are two disjoint (d', C)- and (C, c')-dipaths Q and Q', respectively. Then the first vertex x in G_i along Q is in X and the last vertex y in G_i along Q' is in Y. Since Q and Q' are disjoint, x and y are distinct.

(ii) Suppose that there are vertices $x \in X$ and $y \in Y$, where $x \neq y$. Let Q be a (d', G_i)-dipath with terminal vertex x and let Q' be a (G_i, c')-dipath with initial vertex y. Observe that Q and Q' do not intersect because G_i is a strong component of $D - \{c', d'\}$. Since G_i is strong, there are an (x, y)-dipath R and a $(y, R - y)$-dipath R' in G_i. Now P contains a (Q, Q')-dipath P'. Thus $P' \cup Q[s(P'), x] \cup Q'[y, t(P')] \cup R \cup R'$ is an F_3-subdivision. $\lozenge$

Case 2: Assume now there is no (d', c')-dipath in $D - G_i$.

If D contains a (c', d', G_i)-suitable F_3-subdivision S, then the two independent (d', c')-dipaths in S intersect G_i. Since G_i is a strong component of $D - \{c', d'\}$, each of these two paths consists of three segments: a (d', G_i)-dipath, followed by a dipath in G_i, and ending with a (G_i, c')-dipath. The idea is to guess which are the first vertices d_1, d_2 and last vertices c_1, c_2 in G_i along these dipaths.

Hence, consider every pair of sets of two vertices, $\{c_1, c_2\}, \{d_1, d_2\}$, where $c_1 \neq c_2$ and $d_1 \neq d_2$. Observe that there must be two independent $(d', \{d_1, d_2\})$-dipaths whose internal vertices are not in G_i and two independent $(\{c_1, c_2\}, c')$-dipaths whose internal vertices are not in G_i. We can check for the existence of two independent $(d', \{d_1, d_2\})$-dipaths with internal vertices not in G_i by running a Menger algorithm in the digraph obtained from D by deleting all the arcs with tail in G_i. Similarly, we also check the existence of two independent $(\{c_1, c_2\}, c')$-dipaths with internal vertices not in G_i. If one of these pairs of dipaths do not exist, then we proceed to the next pair. If the two pairs of dipaths exist, they are internally disjoint from each other since G_i is a strong component. In that case, one can easily see that D contains a (c', d', G_i)-suitable F_3-subdivision S such that d_1, d_2 (resp. c_1, c_2) are the first (resp. last) vertices in G_i along the two independent (d', c')-dipaths in S if and only if the digraph $\hat{G}_i(\{c_1, c_2\}, \{d_1, d_2\})$ has no (c'', d'')-*forced* F_3-subdivision. Henceforth, we run F_3-`Strong+`$(G_i, \{c_1, c_2\}, \{d_1, d_2\})$. If this procedure returns 'yes', we also return 'yes'. If it returns 'no', we proceed to the next pair of sets $\{c_1, c_2\}, \{d_1, d_2\}$.

If all the pairs have been considered without returning 'yes', we return 'no'. This procedure is clearly valid provided that we have F_3-`Strong+` subroutine.

Hence our algorithm is valid and runs in polynomial time provided that the procedure F_3-`Strong+` is valid and runs in polynomial time. We now describe this subprocedure.

Detecting F_3 in strong digraphs We now present procedure F_3-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$. Recall that procedure F_3-`Strong+`$(D, \{c_1, c_2\}, \{d_1, d_2\})$ returns 'yes' if it finds an F_3-subdivision in $\hat{D}$, and should returns 'no' if $\hat{D}$ has no (c'', d'')-forced F_3-subdivision. The assumption is that the input digraph D is strongly connected.

In the first phase, we treat the case when D is not 2-connected and reduce to the case when it is. Suppose that D has a cutvertex x. Let $X_1, \dots, X_p$ be the connected components of $D - x$, and for $1 \leq i \leq p$, let $D_i = D[X_i \cup \{x\}]$. Observe that each D_i is strong because D is strong.

Suppose first that c_1 and c_2 lie in different connected components of $D - x$, say X_1 and X_2 (respectively). Let P_1 be an (x, c_1)-dipath in D_1, P'_1 a $(c_1, P_1 - c_1)$-dipath, and P_2 a (c_2, x)-dipath in D_2. The digraph $P_1 \cup P'_1 \cup P_2 \cup c_1c'' \cup c_2c''$ is an F_3-subdivision in $\hat{D}$, and we return 'yes'. Similarly, if d_1 and d_2 are in different connected components of $D - x$, $\hat{D}$ contains an F_3-subdivision, and we return 'yes'.

Henceforth, we may assume that there is $i, j \in \{1, \dots, p\}$ such that $\{c_1, c_2\} \subseteq V(D_i)$ and $\{d_1, d_2\} \subseteq V(D_j)$. If $i \neq j$, then in D, there cannot be two internally disjoint (d'', c'')-dipaths, and thus there is no (c'', d'')-forced F_3-subdivision in $\hat{D}$. Therefore, we return 'no'. If $i = j$, then since F_3 is 2-connected, there is a (c'', d'')-forced F_3-subdivision in $\hat{D}$ if and only if there is a (c'', d'')-forced F_3-subdivision in $\hat{D}_i(\{c_1, c_2\}, \{d_1, d_2\})$. Hence we return F_3-`Strong+`$(D_i, \{c_1, c_2\}, \{d_1, d_2\})$.

Assume now that D is 2-connected, and so D is robust. The procedure uses a similar approach as the procedure F_2-`Subdivision`(D) to decide whether a digraph D contains an F_2-subdivision, and a key notion is the one of opposite dipaths. Recall that two dipaths are (D_1, D_2)-*opposite* if they are disjoint and one of them is a (D_1, D_2)-dipath and the other is a (D_2, D_1)-dipath. Since an F_2-subdivision contains an F_3-subdivision, Lemma 48 implies directly the following one.

Lemma 50. *Let D be a digraph and D_1 and D_2 disjoint non-trivial strong subdigraphs of D. If there are (D_1, D_2)-opposite paths in D, then D contains an F_3-subdivision.*

Lemma 51. *Suppose that D' is a strong subdigraph of D and R is a path in D with its endvertices in D' and with its internal vertices in $D - D'$. If the path R has three blocks, then D contains an F_3-subdivision.*

Proof. Let $s = s(R)$, $t = t(R)$. Let Q be a (t, s)-dipath in D' and let Q' be an $(s, Q - s)$-dipath in D'. Then $Q \cup Q' \cup R$ is an F_3-subdivision in D. □

Returning to the algorithm description, we first check if there are two disjoint directed cycles in D. If not, then one can solve the problem in polynomial time according to Theorem 44.

If two such cycles C_1 and C_2 exist, then we first compute the strong component D_1 of C_1 in $D - C_2$, and next we compute the strong component D_2 of C_2 in $D - D_1$. Hence D_1 and D_2 are two disjoint non-trivial strong subdigraphs in D. Moreover they satisfy the following property (Claim 48.1).

Claim 51.1. *If P is a (D_1, D_2)-dipath and Q is a (D_2, D_1)-dipath, then P and Q are internally disjoint.*

We check if there are (D_1, D_2)-opposite paths in D. If there are, then by Lemma 50, D contains an F_3-subdivision and we return 'yes'. Henceforth we may assume that there are no (D_1, D_2)-opposite paths in D.

By Lemma 47, there is a directed ear P_1 of D_1. Since $D_1 \cup P_1$ is strong, P_1 must intersect D_2. Furthermore, the intersection of P_1 and D_2 is reduced to a single vertex, because there are no (D_1, D_2)-opposite paths. Let u_1 be the initial vertex of P_1, v_1 the terminal vertex of P_1, and let u_2 be the vertex of $P_1 \cap D_2$. By Lemma 47, there is a directed ear P_2 of D_2. If the terminal vertex of P_2 is u_2, then we consider the converse of $\hat{D}$, P_1 and P_2 and exchange the roles of c'' and d'' (i.e. $(c'', d'') := (d'', c'')$) and their neighbours $((\{c_1, c_2\}, \{d_1, d_2\}) := (\{d_1, d_2\}, \{c_1, c_2\})$. (This is valid since F_3 is self-converse.) Hence, we may assume that the terminal vertex v_2 of P_2 is different from u_2. Similarly to P_1, the dipath P_2 intersects D_1 in a single vertex w_1. Clearly, $w_1 = v_1$ for otherwise $P_1[u_2, v_1]$ and $P_2[w_1, v_2]$ are (D_1, D_2)-opposite paths. Furthermore, $P_1[u_1, u_2]$ and $P_2[v_1, v_2]$ are disjoint for otherwise, there are two (D_1, D_2)-opposite dipaths.

Set $P = P_1P_2[v_1, v_2]$. We check whether $D - (D_1 \cup D_2)$ contains a non-trivial strong component D_3. If D_3 does not intersect $P[u_1, v_1]$, then we replace D_1 and D_2 by two disjoint non-trivial strong digraphs, $D_1 \cup D_2 \cup P[u_1, v_1]$ and D_3, respectively. Similarly, if D_3 does not intersect $P[u_2, v_2]$, then we replace D_1, D_2 by disjoint non-trivial strong digraphs $D_1 \cup D_2 \cup P[u_2, v_2]$ and D_3. In either case, we extend the first digraph to a strong component of $D - D_3$, while D_3 is already a strong component in the complement of the first digraph. Thus, Claim 51.1 remains valid. Observe that this change makes the order of D_1 increase. We also redefine u_1, v_1, u_2, v_2 and the path P if the change occurred.

If D_3 intersects both $P[u_1, v_1]$ and $P[u_2, v_2]$, then we return 'yes'. This is valid because in this case, by Claim 48.2, D contains an F_2-subdivision and so an F_3-subdivision.

Henceforth, we may assume that all strong components of $D - (D_1 \cup D_2)$ are trivial, that is $D - (D_1 \cup D_2)$ is acyclic.

Let F_{12} be the set of pairs $(x, y) \in V(D_1) \times V(D_2)$ such that there exists a (D_1, D_2)-dipath R with $s(R) = x$ and $t(R) = y$. Similarly, let F_{21} be the set of pairs $(y, x) \in V(D_2) \times V(D_1)$ such that there exists a (D_2, D_1)-dipath R with $s(R) = y$ and $t(R) = x$. By Claim 51.1 and because there are no (D_1, D_2)-opposite dipaths, we have one of the following two possible outcomes:

Case (A): $F_{21} = \{(u_2, v_1)\}$. In this case, $F_{12} = (U_1 \times \{u_2\}) \cup (\{v_1\} \times V_2)$, where $\{u_1\} \subseteq U_1 \subseteq V(D_1)$ and $\{v_2\} \subseteq V_2 \subseteq V(D_2)$.

Case (B): $F_{21} = \{(u_2, v_1), (v_2, u_1)\}$. In this case, $F_{12} = \{(u_1, u_2), (v_1, v_2)\}$. By setting $U_1 = \{u_1\}$ and $V_2 = \{v_2\}$, the set F_{12} can be written in the same way as in Case (A).

For each vertex $x \in V(D) \setminus V(D_1 \cup D_2)$, there is an $(x, D_1 \cup D_2)$-dipath and a $(D_1 \cup D_2, x)$-dipath. Since $D - (D_1 \cup D_2)$ has only trivial strong components, these two paths are internally disjoint and form a (D_1, D_2)-path R_x. We define $Z(U_1, u_2)$ as the set of all vertices $x \in V(D) \setminus V(D_1 \cup D_2)$, whose path R_x is a (U_1, u_2)-dipath. In the same way we define vertex-sets $Z(u_2, v_1)$, $Z(v_1, V_2)$, and $Z(v_2, u_1)$.

Note that the latter set may be non-empty only when we have Case (B) and that these four sets partition $V(D) \setminus V(D_1 \cup D_2)$.

Next, we derive a sufficient condition for existence of F_3-subdivisions in $\hat{D}$.

Claim 51.2. *If there is a (D_1, D_2)-path with two or three blocks in $\hat{D}$, then $\hat{D}$ contains an F_3-subdivision.*

Subproof. Let R be a (D_1, D_2)-inpath with two blocks and let y be the vertex of outdegree 2 in R. Let w_1 be a vertex in $\{u_1, v_1\} \setminus \{s(R)\}$ and let $w_2 = u_2$ if $w_1 = u_1$ and $w_2 = v_2$ if $w_1 = v_1$. If $P[w_1, w_2]$ is disjoint from R, then the path $R \cup P[w_1, w_2] \cup D_2[w_2, t(R)]$ has three blocks and by Lemma 51, $\hat{D}$ contains an F_3-subdivision. On the other hand, if $P[w_1, w_2]$ intersects R, let z be the first vertex on $P[w_1, w_2]$ that lies on R. Since D_1 is a strong component of $D - D_2$, $z \in R[y, t(R)]$ and $z \neq y$. Therefore, $R[s(R), z]$ and $P[w_1, z]$ form a path with three blocks and we are done by Lemma 51.

Similarly, by directional duality, if there is a (D_1, D_2)-outpath with two blocks, then $\hat{D}$ contains an F_3-subdivision.

Suppose now that there is a (D_1, D_2)-outpath R in $\hat{D}$ with three blocks. Let $s = s(R)$, $t = t(R)$, and let x and y be vertices on R whose indegree and outdegree (respectively) is equal to 2. $R[s, x] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$ because D_1 is a strong component of $D - D_2$ If $R[x, t] \cap (Z(u_2, v_1) \cup Z(v_2, u_1)) = \emptyset$, then there is a (D_1, D_2)-path with two blocks and we have the result by the above case.

Hence, we may assume that R does not intersect $(Z(u_2, v_1) \cup Z(v_2, u_1)$. In particular, R and $P[U_2, v_1]$ are internally disjoint. If $s \neq v_1$, then $RD[t, u_2]P[u_2, v_1]$ is an ear of D_1 with three blocks and by Lemma 51 D contains an F_3-subdivision. Similarly, if $t \neq u_2$, then D contains an F_3-subdivision. Henceforth we may assume that $s = v_1$ and $t = u_2$.

If R is internally disjoint from $P[u_1, u_2]$, then $P[u_1, v_1] \cup D_1[v_1, u_1] \cup R$ is an F_3-subdivision in $\hat{D}$, and if R is internally disjoint from $P[v_1, v_2]$, then $P[u_2, v_2] \cup D_2[v_2, u_2] \cup R$ is an F_3-subdivision in $\hat{D}$. Thus, we may assume that R intersects both $P]u_1, u_2[$ and $P]v_1, v_2[$.

Let z be the first vertex on P that belongs to R. If $z \in R[s, y[$, then $P[u_1, z]R[z, t]$ is a (D_1, D_2)-dipath with three blocks, and we get the result as above because its initial vertex is not v_1. If $z \in R]y, t]$, then $P[u_1, z] \cup R \cup P[u_2, v_1] \cup D[v_1, u_1]$ is an F_3-subdivision. Therefore, we may assume that $z = y$.

Analogously, we may assume that the last vertex on $P[v_1, v_2]$ that belongs to R is x. Now $P[u_1, y]R[y, x]P[X, v_2]$ contains a (D_1, D_2)-dipath with initial vertex u_1 and terminal vertex v_2. By Claim 51.1, this dipath and $P[u_2, v_1]$ are (D_1, D_2)-opposite dipaths, a contradiction.

Similarly, by directional duality, if there is a (D_1, D_2)-inpath with three blocks, then $\hat{D}$ contains an F_3-subdivision. This completes the proof. $\Diamond$

Claim 51.3. (i) *For $i = 1, 2$, if $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$ and $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$, then $\hat{D}$ contains an F_3-subdivision.*

(ii) *For $i = 1, 2$, if $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$ and $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$, then $\hat{D}$ contains an F_3-subdivision.*

(iii) *If $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ or $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$, then $\hat{D}$ contains an F_3-subdivision.*

Subproof. (i) Suppose $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$ and $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$. Without loss of generality, we may assume that $i = 1$ and that $c_1 \in V(D_1)$ and $c_2 \notin V(D_1)$. Since D is strong, there is a $(D_1 \cup D_2, c_2)$-dipath Q and a $(c_2, D_1 \cup D_2)$-dipath R in D. If $s(Q) \in D_2$, then $c_1 c'' c_2 \overleftarrow{Q}$ is a (D_1, D_2)-path in $\hat{D}$ with two blocks, so by Claim 51.2, $\hat{D}$ contains an F_3-subdivision. If $s(Q) \in D_1$, then $t(R) \in D_1$, because D_1 is a strong component in $D - D_2$. Hence $c_1 c'' c_2 R$ is a (D_1, D_2)-path with two blocks, so by Claim 51.2, $\hat{D}$ contains an F_3-subdivision.

(ii) This claim is proved analogously to (i).

(iii) Assume that $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$. (The case when $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ is proved in the same way.) Since D is strong, there exist a $(D_1 \cup D_2, c_1)$-dipath Q_1, and a $(D_1 \cup D_2, c_2)$-dipath Q_2. If $s(Q_1)$ and $s(Q_2)$ do not lie in the same D_i, then there is a (D_1, D_2)-path (which is either contained in $Q_1 \cup Q_2$ if Q_1 and Q_2 intersect, or passes through c'' if they are disjoint) in $\hat{D}$ having two blocks, so by Claim 51.2, $\hat{D}$ contains an F_3-subdivision. Henceforth, we may assume that $s(Q_1)$ and $s(Q_2)$ are in the same D_i, say D_1.

Since D is strong, for $i = 1, 2$, there exists a $(c_i, D_1 \cup D_2,)$-dipath R_i. Its endvertex $t(R_i)$ cannot be in D_1, because D_1 is a strong component of $D - D_2$. Thus $t(R_i) \in V(D_2)$. If R_1 intersects Q_2 and R_2 intersects Q_2, then c_1 and c_2 are in the same strong component of $D - (D_1 \cup D_2)$, which contradicts one of our previous assumptions. Therefore, without loss of generality, we may assume that R_2 does not intersect Q_1. Now $Q_1 \cup c_1c'' \cup c''c_2 \cup R_2$ is a (D_1, D_2)-path with three blocks. Thus by Claim 51.2, $\hat{D}$ contains an F_3-subdivision. $\Diamond$

In view of Claim 51.3, if $\{c_1, c_2\} \cap V(D_i) \neq \emptyset$ and $\{c_1, c_2\} \setminus V(D_i) \neq \emptyset$ or $\{d_1, d_2\} \cap V(D_i) \neq \emptyset$ and $\{d_1, d_2\} \setminus V(D_i) \neq \emptyset$ for some $i \in \{1, 2\}$, then we return 'yes'. The same holds if $\{c_1, c_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$ or $\{d_1, d_2\} \subseteq V(D) \setminus (V(D_1) \cup V(D_2))$. Thus, we may assume henceforth that there are indices $i_c, i_d \in \{1, 2\}$ such that $\{c_1, c_2\} \subseteq V(D_{i_c})$ and $\{d_1, d_2\} \subseteq V(D_{i_d})$.

We now run a procedure $\texttt{2or3blocks}(D, D_1, D_2)$ for finding a (D_1, D_2)-path with two or three blocks in D. If such a path is found, we stop the main procedure $F_3\texttt{-strong+}$ by returning 'yes' since in this case we have an F_3-subdivision by Claim 51.2.

The procedure $\texttt{2or3blocks}(D, D_1, D_2)$ proceeds as follows. Let S_1^- be the insection of D_1 in $D - D_2$ and S_2^- the insection of D_2 in $D - D_1$. It is easy to see that there is a (D_1, D_2)-inpath with two blocks, if and only if $S_1^- \cap S_2^-$ contains a vertex of $D - (D_1 \cup D_2)$. Therefore we compute $S_1^- \cap S_2^-$ and return 'yes' if this set contains a vertex in $D - (D_1 \cup D_2)$.

Similarly, to detect if there is a (D_1, D_2)-outpath with two blocks, we compute the outsection S_1^+ of D_1 in $D - D_2$ and the outsection S_2^+ of D_2 in $D - D_1$, and if $S_1^+ \cap S_2^+$ contains a vertex of $D - (D_1 \cup D_2)$, we return 'yes'.

Let us now describe how to discover paths with three blocks. Let tz be an arc and y be a vertex in D such that $t \in V(D - D_2)$, $y, z \in V(D - (D_1 \cup D_2))$, $y \notin \{t, z\}$. Arc tz and vertex y are said to be in 3-*block-position* if there are a (y, z)-dipath and a (y, D_2)-dipath in $D - (V(D_1) \cup \{t\})$ which are independent, and a (D_1, t)-dipath in $D - (D_2 \cup \{y, z\})$.

Claim 51.4. *There is a (D_1, D_2)-outpath in D with three blocks if and only if there are an arc tz and a vertex y in 3-block-position.*

Subproof. Trivially, if there is a (D_1, D_2)-outpath with three blocks, then there are an arc tz and a vertex y in 3-block-position.

Let us now prove the converse. Assume that tz and y are in 3-block-position. Let Q_1 and Q_2 be the two independent paths from y to z and D_2, respectively, and let R be the (D_1, t)-dipath in $D - (D_2 \cup \{y, z\})$.

If R does not intersect $Q_1 \cup Q_2$, then $R \cup tz \cup \overleftarrow{Q_1} \cup Q_2$ is a (D_1, D_2)-path with three blocks.

Assume now that R intersects $Q_1 \cup Q_2$. Let x be the first vertex along R in $V(Q_1 \cup Q_2)$. Note that $x \neq y$ by definition of R. If $x \in V(Q_1)$, then $R[s(R), x] \cup \overleftarrow{Q_1}[x, y] \cup Q_2$ is a (D_1, D_2)-path with three blocks. If $x \in V(Q_2)$, then consider a $(z, D_1 \cup D_2)$-dipath R' in D. Because D_1 is a strong component of $D - D_2$ and $RtzR'$ is a dipath, $t(R') \in D_2$. Furthermore R' does not meet $R \cup Q_1 \cup Q_2[y, x]$, because $D - (D_1 \cup D_2)$ is acyclic. Hence $R[s(R), x] \cup \overleftarrow{Q_2}[x, y] \cup Q_1 \cup R'$ is (D_1, D_2)-path with three blocks. $\Diamond$

Therefore, for every possible arc tz and vertex y such that $t \in V(D - D_2)$, $y, z \in V(D - (D_1 \cup D_2))$, $y \notin \{t, z\}$, we check if they are in 3-block-position. This can be done by running Menger algorithm. If we find an arc and a vertex in 3-block-position, then we return 'yes' because there is an F_3-subdivision by Claims 51.4 and 51.2.

We deal similarly with the (D_1, D_2)-inpaths with three blocks. This ends the procedure `2or3blocks`-(D, D_1, D_2). After it, there is no (D_1, D_2)-path in $\hat{D}$ with two blocks and no (D_1, D_2)-path in D with three blocks.

We now show that we can reduce D to a digraph with vertex set $V(D_1) \cup V(D_2)$.

Let D^* be the digraph obtained from $D_1 \cup D_2$ by adding all arcs in $F_{12} \cup F_{21}$. In other words, we add all arcs x_1x_2 with $x_1 \in V(D_1)$ and $x_2 \in V(D_2)$ such that there is a (D_1, D_2)-dipath with initial vertex x_1 and terminal vertex x_2, and adding all arcs x_2x_1 with $x_1 \in V(D_1)$ and $x_2 \in V(D_2)$ for which there is a (D_2, D_1)-dipath with initial vertex x_2 and terminal vertex x_1. Set $\hat{D}^* = \hat{D}^*(\{c_1, c_2\}, \{d_1, d_2\})$.

Claim 51.5. *$\hat{D}$ contains a (c'', d'')-forced F_3-subdivision if and only if $\hat{D}^*$ contains a (c'', d'')-forced F_3-subdivision.*

Subproof. As mentioned above, D^* either contains the four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ (Case (B)), or contains the arcs uu_2 ($u \in U_1$), u_2v_1, and v_1v ($v \in V_2$), which is Case (A). For each of these arcs uv, there is a corresponding directed path R_{uv} in D. One can transform a (c'', d'')-forced F_3-subdivision S^* in $\hat{D}^*$ into an F_3-subdivision S of $\hat{D}$ by replacing each arc uv in S^* between D_1 and D_2 by the path R_{uv}. If all added paths R_{uv} are pairwise internally disjoint, this clearly gives rise to an F_3-subdivision in $\hat{D}$. The only possibility that two of such paths may not be internally disjoint (cf. Lemma 50) is that we have two paths R_{uu_2} and $R_{u'u_2}$ (where $u, u' \in U_1$ and $u \neq u'$) or two paths R_{v_1v} and $R_{v_1v'}$ (where $v, v' \in V_2$ and $v \neq v'$). However, since every vertex in F_3 has in- and outdegrees at most 2, there are at most two such paths entering u_2 and at most two leaving v_1. For two of them, we can always achieve that their intersection is a common subpath, and in that case, the resulting digraph is again an F_3-subdivision. Clearly, the resulting F_3-subdivision in $\hat{D}$ is (c'', d'')-forced.

Suppose now that $\hat{D}$ contains a (c'', d'')-forced F_3-subdivision S. Let a', b', be the vertices of S corresponding to a, b, respectively, and let C be the directed cycle in S. If $a', b' \in V(D_1) \cup V(D_2)$, then the arcs in S that are not in $D_1 \cup D_2$ form a collection of internally disjoint (D_1, D_2)- and (D_2, D_1)-dipaths. By replacing each of these dipaths by the corresponding arc in D^*, we obtain an F_3-subdivision in $\hat{D}^*$.

Assume now that $a' \notin V(D_1) \cup V(D_2)$. The cycle C must intersect both D_1 and D_2, and thus C contains a (D_2, D_1)-dipath. Without loss of generality, we may assume that the initial vertex of this dipath is u_2 and its terminal vertex is v_1.

Now let z_2 be the first vertex in $V(D_1 \cup D_2)$ along the (a', c'')-dipath in S. This vertex exists because $\{c_1, c_2\} \subseteq V(D_1 \cup D_2)$. Now since there are no (D_1, D_2)-opposite paths, and by definition of the D_i, $z_2 \in V(D_2)$, so a' does not lie in $C[u_2, v_1]$. Let y_2 be the first vertex after a' along C in $V(D_1 \cup D_2)$. For the same reason, $y_2 \in V(D_2)$ and so v_1 is the unique vertex in $C \cap D_1$, for otherwise, there would be (D_1, D_2)-opposite paths. Note that $y_2, z_2 \in V_2$.

If $\{c_1, c_2\} \subseteq V(D_1)$, then the (z_2, c'')-dipath in S contains a (D_2, D_1)-dipath that together with $C[v_1, y_2]$ gives (D_1, D_2)-opposite paths.

This shows that $\{c_1, c_2\} \subseteq V(D_2)$. The (d'', c'')-dipath in $S - E(C)$ must have all its internal vertices in D_2, because every (D_1, D_2)-dipath meets $\{u_2, v_1\}$. Therefore the digraph obtained from $S \cap (D_1 \cup D_2)$ by adding the arcs u_2v_1, v_1y_2, and v_1z_2 is an F_3-subdivision in $\hat{D}^*$.

If $b' \notin V(D_1) \cup V(D_2)$, we get the result analogously. $\Diamond$

In view of Claim 51.5, we replace D by D^*. Henceforth, now $V(D) = V(D_1) \cup V(D_2)$. Moreover, there are at most two arcs with tail in D_2 and head in D_1, namely u_2v_1 and possibly v_2u_1.

Let D'_1 be the digraph obtained from D by contracting D_2 into a vertex z_2. If all four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ are present in D (Case (B)), then we also add into D'_1 the arcs u_1v_1 and v_1u_1 if they are not already contained in D_1. Similarly, we let D'_2 be the digraph obtained from D by contracting D_1 into a vertex z_1 and adding the arcs u_2v_2 and v_2u_2 if D contains all four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$. Observe that D'_1 and D'_2 are both strong, and contain fewer vertices than D.

If $i_c = i_d$, then we return F_3-Strong+$(D'_{i_c}, \{c_1, c_2\}, \{d_1, d_2\})$. This is valid by the following claim.

Claim 51.6. *If $i_c = i_d$, then $\hat{D}$ contains a (c'', d'')-forced F_3-subdivision if and only if $\hat{D}_{i_c}$ contains a (c'', d'')-forced F_3-subdivision.*

Subproof. We shall assume that $i_c = 1$. (The case when $i_c = 2$ is proved in the same way.) Suppose first that $\hat{D}$ contains a (c'', d'')-forced F_3-subdivision S. If S does not intersect D_2, then S is a (c'', d'')-forced F_3-subdivision in $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$. Henceforth we assume that S intersects D_2. Observe that vertices a' and b' corresponding to a and b in S belong to a (d'', c'')-dipath in S. Therefore, a' and b' belong to D_1, since there are no (D_1, D_2)-opposite dipaths in D. Consequently, every vertex in S lies on a (d'', c'')-dipath or on an (a', b')-dipath in S, and each such path intersects D_2 in at most one vertex. If S contains only one vertex v in D_2, the digraph S' obtained from S by replacing v by z_2, we obtain a (c'', d'')-forced F_3-subdivision in $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$. If S contains two vertices in D_2, then S contains all four arcs $u_1u_2, u_2v_1, v_1v_2, v_2u_1$ and hence D'_1 contains the arcs u_1v_1 and v_1u_1. Moreover, $\{a', b'\} = \{u_1, v_1\}$ and the cycle in S is the 4-cycle $u_1u_2v_1v_2u_1$. Therefore, the arcs u_1v_1 and v_1u_1 are not both in S. Then we replace the cycle in S by the cycle $u_1v_1z_2u_1$ in D'_1 and obtain a (c'', d'')-forced F_3-subdivision in $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$.

Suppose now that $\hat{D}'_1(\{c_1, c_2\}, \{d_1, d_2\})$, contains a (c'', d'')-forced F_3-subdivision S'. Let us first assume that S' does not contain z_2. If S' contains an arc that is not in $A(D)$, then this is either the arc $e = u_1v_1$ or $e' = v_1u_1$. This arc was added to D'_1 only in Case (B). Thus, we can replace e in S' by the path $u_1u_2v_1$ and e' by the path $v_1v_2u_1$. By making these changes (if needed), we obtain a (c'', d'')-forced F_3-subdivision in $\hat{D}$. Henceforth we assume that S contains z_2.

If S' contains an arc $e \in \{v_1u_1, u_1, v_1\}$ and $e \notin A(D)$, then we have Case (B). In S', the vertex z_2 has in- and outdegree equal to 1, while each of u_1 and v_1 has either in- or outdegree equal to 2. Therefore, the arc in $\{v_1u_1, u_1, v_1\} \setminus \{e\}$ cannot be in S'. By replacing the path in S' joining u_1 and v_1 through z_2 by that arc, we obtain an F_3-subdivision that does not contain z_2, and we are done in the same way as above. Thus, we may assume that S' contains no edge in $\{v_1u_1, u_1, v_1\} \setminus A(D)$.

The vertex z_2 has an inneighbour x_1 and an outneighbour y_1 in S', and possibly has a third neighbour z_1. If z_1 exists, then we assume that the arcs x_1z_2 and z_2y_1 lie on the cycle in S'. By definition of contraction, x_1 has an outneighbour x_2 in $V(D_2)$ and y_1 has an inneighbour y_2 in $V(D_2)$. Moreover if z_1 exists, let w_2 be one of its outneighbours (resp. inneighbours) in $V(D_2)$ corresponding to the arc joining z_1 and z_2 in D'_1. Let Q be an (x_2, y_2)-dipath in D_2 and Q' be a (w_2, Q)-dipath (resp. (Q, w_2)-dipath) in D_2 if $z_2z_1 \in A(S')$ (resp. $z_1z_2 \in A(S')$). Now the digraph obtained from S' by replacing z_2 and the arcs incident to it by the paths Q and Q', and the arcs x_1x_2, y_1y_2 and z_1w_2 or w_2z_1, is a (c'', d'')-forced F_3-subdivision in $\hat{D}$. $\Diamond$

Henceforth, we have $i_c \neq i_d$. If $i_d = 2$ and $i_c = 1$, then a (c'', d'')-forced F_3-subdivision contains two disjoint arcs from D_2 to D_1. Thus, necessarily v_2u_1 is an arc, because there are no (D_1, D_2)-opposite paths. In this case, we consider exchanging the roles of D_1 and D_2. Thus, we may assume henceforth that we are in the case when $i_d = 1$ and $i_c = 2$.

Let $D_1^* = D[V(D_1) \cup \{u_2\}]$ and $D_2^* := D[V(D_2) \cup \{v_1\}]$. Observe that D_1^* and D_2^* are both strong.

A (c'', d'')-forced F_3-subdivision contains two internally disjoint (d'', c'')-dipaths. Therefore, using a Menger algorithm, we check if two such dipaths exist in $\hat{D}$. If two such dipaths do not exist, then we return 'no'. Otherwise there are two internally disjoint (d'', c'')-dipaths. Because there are no (D_1, D_2)-opposite paths, one of them say P_1^* must go through v_1 and the other, say P_2^*, through u_2. We return (F_3-Strong+$(D_1^*, \{v_1, u_2\}, \{d_1, d_2\})$ or F_3-Strong+$(D_2^*, \{c_1, c_2\}, \{v_1, u_2\})$. This is valid by the following claim.

Claim 51.7. *$\hat{D}$ contains a (c'', d'')-forced F_3-subdivision if and only if either $\hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$ or $\hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$ contains a (c'', d'')-forced F_3-subdivision.*

Subproof. Set $\hat{D}_1^* := \hat{D}_1^*(\{v_1, u_2\}, \{d_1, d_2\})$ and $\hat{D}_2^* := \hat{D}_2^*(\{c_1, c_2\}, \{v_1, u_2\})$.

A (c'', d'')-forced F_3-subdivision in $\hat{D}_1^*$ (or one in $\hat{D}_2^*$) can easily be transformed into a (c'', d'')-forced F_3-subdivision in $\hat{D}$ by replacing the arcs v_1c'' and u_2c'' by $P_1^*[v_1, c'']$ and $P_2^*[u_2, c'']$.

Suppose now that $\hat{D}$ contains a (c'', d'')-forced F_3-subdivision S. Let C be the directed cycle in S and let Q_1 and Q_2 be the two internally disjoint (d'', c'')-dipaths in S. Because there are no (D_1, D_2)-opposite paths, one of these dipaths, say Q_1 goes through v_1 and the other goes through u_2. Moreover, C intersects either D_1 or D_2 in at most one vertex. If D does not contain the arc v_2u_1 (Case (A)), then if C intersects D_1 (resp. D_2) in one vertex, this vertex must be v_1 (resp. u_2). We may assume that the same holds in Case (B) after possibly exchanging the roles of u_1 and v_1 and of u_2 and v_2. Hence the digraph obtained from S by replacing $Q_1[d'', v_1]$ and $Q_2[d'', u_2]$ (resp. $Q_1[v_1, c'']$ and $Q_2[u_2, c''])$ by the arcs $d''v_1$ and $d''u_2$ (resp. v_1c'' and u_2c'') is a (c'', d'')-forced F_3-subdivision in $\hat{D}_2^*$ (resp. $\hat{D}_1^*$). $\Diamond$

This completes the procedure F_3-`Strong+`. Let us now examine its time complexity. Let $T(n)$ be the maximum running time an a digraph with at most n vertices. Clearly, the running time between two recursive calls is bounded by a polynomial $P(n)$. When treating a graph D on n vertices, it then makes a recursive call either to a smaller digraph, or to two smaller digraphs D_1^* and D_2^* such that $|D_1^*| + |D_2^*| \leq n + 2$. Hence $T(n)$ satisfies the inequality

$$T(n) \leq P(n) + \max\{T(n-1)\ ;\ \max\{T(n_1) + T(n_2) \mid n_1 + n_2 \leq n+2, n_1 < n, n_2 < n\}\}.$$

This implies that $T(n)$ is bounded above by a polynomial value in n.

A.2.5 Detecting F_4

Proposition 52. *The digraph F_4 depicted in Figure 8 is tractable. More precisely, F_4-*SUBDIVISION *can be solved in time $O(n^2m(n+m))$.*

Proof. Let D be a digraph. Observe that every F_4-subdivision contains an F_4-subdivision in which the arc cd is not subdivided. Henceforth by F_4-subdivision, we mean such a subdivision.

Given four distinct vertices a', b', c', d' such that $c'd'$ is an arc, we say that an F_4-subdivision is $(a', b', c'd')$*-forced* if a' is its a-vertex, b' its b-vertex, c' its c-vertex, and d' its d-vertex.

We shall present a procedure F_4-`Subdivision`$(D, a', b', c'd')$, that returns 'no' only if there is no $(a', b', c'd')$-forced F_4-subdivision in D, and returns 'yes' if it finds an F_4-subdivision in D (not necessarily one that is $(a', b', c'd')$-forced). We proceed as follows.

Suppose first that $a'b'$ is an arc. Using a Menger algorithm, we check whether there are two independent $(b', \{a', c'\})$-dipaths in $D - d'$, and using a search, we check whether there exists a (c', a')-dipath in $D' - \{b', d'\}$. If three such dipaths do not exist, then there is no $(a', b', c'd')$-forced F_4-subdivision in D, and we return 'no'. If three such dipaths exist, then we return 'yes'. This is valid by virtue of the following claim.

Claim 52.1. *If there are a (c', a')-dipath R in $D' - \{b', d'\}$ and two independent $(b', \{a', c'\})$-dipaths P_1, P_2 in $D - d'$, then D contains an F_4-subdivision.*

Subproof. Without loss of generality, we may assume that $t(P_1) = a'$ and $t(P_2) = c'$. The dipath R contains a subdipath R' with initial vertex s in P_2 and terminal vertex in $P_1]b', a']$. Let s^+ be the outneighbour of s in $P_2 \cup c'd'$. Then $a'b' \cup P_1 \cup P_2[b', s] \cup R' \cup ss^+$ is an F_4-subdivision. $\Diamond$

Henceforth, we assume that $a'b'$ is not an arc in D.

If $d^+_{D-\{c',d'\}}(a') = 0$, then there is no (a', b')-dipath in $D - c' - d'$, and thus no $(a', b', c'd')$-forced F_4-subdivision. Hence we return 'no'.

If $d^+_{D-\{c',d'\}}(a') = 1$, then denote by a'' the unique outneighbour of a' in $D - \{c', d'\}$. By our assumption, $a'' \neq b'$. Let D^* be the digraph obtained from D by first removing all arcs entering a'' and then identifying a' and a'' into a single vertex a^*. Note that a^* is dominated by the inneighbours of a' in D and dominates the outneighbours of a'' in D. We the return F_4-$\texttt{Subdivision}(D^*, a^*, b', c'd')$. The validity of this recursive call is shown by the following claim.

Claim 52.2. *If $d^+_{D-\{c',d'\}}(a') = 1$, then D contains an $(a', b', c'd')$-forced F_4-subdivision if and only if D^* contains an $(a^*, b', c'd')$-forced F_4-subdivision.*

Subproof. Assume that S is an $(a', b', c'd')$-forced F_4-subdivision in D. Since $d^+_{D-\{c',d'\}}(a') = 1$, S contains the arc $a'a''$ since a'' is the unique outneighbour of a' in $D' - c' - d'$. Now the digraph S^* obtained from S by replacing a' and a'' and the four arcs $ua', va', a'a'', a''w$ by the vertex a^* and the three arcs ua^*, va^*, a^*w is an $(a^*, b', c'd')$-forced F_4-subdivision in D^*, because $a'' \neq b'$.

Conversely, if S^* is an $(a^*, b', c'd')$-forced F_4-subdivision in D^*, then the digraph S obtained from S^* by replacing a^* and its three incident arcs ua^*, va^*, a^*w by vertices a' and a'' and the four arcs $ua', va', a'a'', a''w$ is clearly an $(a', b', c'd')$-forced F_4-subdivision in D. $\Diamond$

Henceforth, we may assume that $d^+_{D-\{c',d'\}}(a') \geq 2$. Using a Menger algorithm, we check whether there are two independent $(b', \{a', c'\})$-dipaths in $D - d'$, and using a search we check whether there exists an (a', b')-dipath in $D - \{c', d'\}$, and whether there exists a (c', a')-dipath in $D - \{b', d'\}$. If four such dipaths do not exist, then there is no $(a', b', c'd')$-forced F_4-subdivision in D, and we return 'no'. If four such dipaths exist, then we return 'yes' by virtue of the following claim.

Claim 52.3. *If there are two independent $(b', \{a', c'\})$-dipaths P_1 and P_2 in $D - d'$, an (a', b')-dipath Q in $D - \{c', d'\}$, and a (c', a')-dipath R in $D - \{b', d'\}$, then D contains an F_4-subdivision.*

Subproof. Without loss of generality, we may assume that $t(P_1) = a'$ and $t(P_2) = c'$. Let v be the last vertex along $Q - b'$ that is in $P_1 \cup P_2$. We distinguish two cases according to whether v is on P_1 or P_2.

<u>Case 1:</u> $v \in V(P_1)$. Note that this is in particular the case when Q is internally disjoint from P_1 and P_2. Let C be the directed cycle formed by the union of $P_1[b'v]$ and $Q[v, b']$, let Q' be the (a', C)-subdipath in Q, and let $R' = RQ'$. The directed walk R' contains a subdipath R'' with initial vertex s in $P_2]b', c']$ and terminal vertex t in C. Let s^+ be the outneighbour of s in $P_2 \cup c'd'$. Then $C \cup P_2[b', s] \cup R'' \cup ss^+$ is an F_4-subdivision.

<u>Case 2:</u> $v \in V(P_2)$. Let v^+ be the outneighbour of v in $P_2 \cup c'd'$. The dipath $Q[a', v]$ contains a subdipath Q' with initial vertex u in P_1 and terminal vertex w in $P_2[b', v]$ whose internal vertices are not in $P_1 \cup P_2[b', v]$. Let C' be the directed cycle formed by the union of $P_2[b', v]$ and $Q[v, b']$. If $u \neq a'$, let u^+ be the outneighbour of u in P_1. Then $C' \cup P_1[b', u^+] \cup Q'$ is an F_4-subdivision. Henceforth, we may assume that $u = a'$.

Let u' be the outneighbour of a' in Q'. Now $d^+_{D-\{c',d'\}}(a') \geq 2$ and $a'b'$ is not an arc. Hence, a' has an outneighbour z distinct from b', c', d', and u'.

- If $z \notin V(C' \cup P_1 \cup Q')$, then $C' \cup P_1 \cup Q' \cup a'z$ is an F_4-subdivision.
- If $z \in V(Q')$, then $C' \cup P_1 \cup a'z \cup Q'[z, w] \cup a'u'$ is an F_4-subdivision.
- Assume $z \in V(P_1)$. If $v^+ \notin V(Q')$, then $P_1 \cup a'z \cup Q' \cup P_2[w, v] \cup Q[v, b'] \cup vv^+$ is an F_4-subdivision. If $v^+ \in V(Q')$, then $v^+ \neq d'$ and so $P_2[v^+, c']$ is not an empty dipath. Denote by C'' the directed cycle $P_1[z, a'] \cup a'z$. The dipath $Q[v^+, b'] \cup P_1[b', z]$ contains a $(P_2[v^+, c'], C'' - a')$-dipath Q''. Let s^+ be the outneighbour of $s(Q'')$ in $P_2 \cup c'd'$. Now $C'' \cup Q[a', v^+] \cup P_2[v^+, s(Q'')] \cup Q'' \cup s(Q'')s^+$ is an F_4-subdivision.

- Assume $z \in V(Q]v, b'])$. Then one can replace the (a', b')-dipath Q by $a'zQ[z, b']$. This dipath is internally disjoint from P_1 and P_2, and we get the result by Case 1.

- Assume finally that $z \in V(P_2[b', v])$. If $u' \neq w$, then $C' \cup P_1 \cup a'z \cup a'u'$ is an F_4-subdivision. Henceforth we assume that $u' = w$, so $a'w$ is an arc. Without loss of generality, we may assume that z precedes w along P_2. For $i = 1, 2$, let b_i^+ be the outneighbour of b' in P_i. By the previous assumption, $b_2^+ \neq w$. Let t be the last vertex along $R - a'$ in $V(P_1 \cup P_2 \cup Q[v, b'])$.

  If $t \in V(P_1)$, then one of the two dipaths $P_1[t, a']$ and $R[t, a']$ has length at least 2. Let t^+ be the outneighbour of t in this dipath, and let T be the other dipath. Now $C' \cup P_1[b', t] \cup T \cup a'w \cup tt^+$ is an F_4-subdivision.

  If $t \in V(Q[v, b'] \cup P_2[w, v])$, then $R[t, a'] \cup a'w \cup P_2[w, v] \cup Q[v, b'] \cup P_1 \cup b'b_2^+$ is an F_4-subdivision.

  If $t \in V(P_2[v, c'])$, then $R[t, a'] \cup a'w \cup P_2[w, t] \cup Q[v, b'] \cup P_1 \cup b'b_2^+$ is an F_4-subdivision.

  If $t \in V(P_2[z, w[)$, then $R[t, a'] \cup a'z \cup P_2[z, v] \cup Q[v, b'] \cup vv^+ \cup P_1$ is an F_4-subdivision.

  It remains to consider the case when $t \in V(P_2]b', z[)$. Let t^+ be the outneighbour of t on P_2. Then $P_1 \cup P_2[b', t^+] \cup R[t, a'] \cup a'w \cup P_2[w, v] \cup Q[v, b']$ is an F_4-subdivision in D.

$\Diamond$

One can easily see that the procedure F_4-`Subdivision`$(D, a', b', c'd')$ runs in linear time as it either reduces the problem in constant time (when $d^+(a') = 1$) or runs a Menger algorithm and at most two searches, which can be done in linear time. Running this procedure for the $O(n^2m)$ possible choices of $(a', b', c'd')$, we obtain an algorithm with running time $O(n^2m(n+m))$ that solves F_4-SUBDIVISION. $\square$

Proposition 53. *For every $i \in \{5, 6, 7\}$, the digraph F_i depicted in Figure **??** is tractable.*

Proof. $i = 5$: Observe first that if a digraph contains an F_5-subdivision, then it contains such an F_5-subdivision in which ac and bd are not subdivided. Henceforth, by F_5-subdivision, we mean an F_5-subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs $a'c'$ and $b'd'$, returns 'yes' if it finds an F_5-subdivision and returns 'no' only if there is no F_5-subdivision with a-vertex a', b-vertex b', c-vertex c' and d-vertex d'. Such a subdivision is said to be $(a'c', b'd')$*-forced.*

We check whether, in $D - \{c', d'\}$, there exists an (a', b')-dipath P and a (b', a')-dipath Q. If two such dipaths do not exist, then there is clearly no $(a'c', b'd')$-forced F_5-subdivision, and we return 'no'. If two such paths P and Q exist, then we return 'yes'. Indeed let b'' be the last vertex on $Q - a'$ that is in $P \cap Q$. If $b'' = b'$, then set $d'' = d'$, otherwise let d'' be the successor of b'' on P. Then the union of the directed cycle $P[a', b'']Q[b'', a']$ and the two arcs $a'c'$ and $b''d''$ form an F_5-subdivision.

Applying the above procedure for all possible pairs of distinct arcs $(a'c', b'd')$, one solves F_5-SUBDIVISION in $O(m^2(n+m))$ time.

$i = 6$: The proof is similar to the case $i = 7$, and we leave details to the reader.

$i = 7$: Observe first that if a digraph contains an F_7-subdivision, then it contains an F_7-subdivision in which dc is not subdivided. Henceforth, by F_7-subdivision, we mean an F_7-subdivision of that kind.

Let us describe a procedure that, given two disjoint arcs sa' and $d'c'$, checks whether there is an F_7-subdivision S with a-vertex a', c-vertex c' and d-vertex d', and such that $sa' \in A(S)$. Such a subdivision is said to be $(sa', d'c')$*-forced.*

With a Menger algorithm, we check whether $D - d'$ contains two independent $(a', \{s, c'\})$-dipaths. If two such dipaths do not exist, then there is clearly no $(a'c', b'd')$-forced F_7-subdivision, and we return 'no'. If two such dipaths P and Q exist, then without loss generality $t(P) = s$ and $t(Q) = c$. The union of the directed cycle Psa', the dipath Q and the arc $d'c'$ is an $(sa', d'c')$-forced F_7-subdivision.

Applying the above procedure for all possible pairs of distinct arcs $(sa', d'c')$, one solves F_7-SUBDIVISION in $O(m^2(n + m))$ time. □

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399